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Research Article

Stability of Riemann–Hadamard Fractional Differential Nonlinear System with Delay Riemann –Katugampola

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Abstract
The purpose of this paper is to investigate the finite-time stability of a proposal problem containing a Riemann-Hadamard fractional differential nonlinear control system with delay Riemann-Katugampola fractional-order systems. The inequalities for satisfying the stability depend on Gronwald's general formulation, which constrains the types of fractionals. The necessary and sufficient conditions are presented in different results to support the stability with delay time of solution with nonlinear functions as well as the numerical examples to illustrate all the interesting results.

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1.Introduction

The topic of stability is a hot topic in control theory. On the other hand, one of the main issues in this subject is regulating time delay systems[1],[2],[3],[4],[5] Time delay systems have been used for many years in a wide range of technologies and systems, including electrical, pneumatic, and hydraulic systems, networks, chemical processes, lengthy transmission lines, and others[5]. Delay may be found in many physical and engineering elements of systems in the control context, delay refers to the duration necessary for information to be communicated and the system to respond. Because systems have a limited time to acquire information and react to these delays, these delays are frequently employed to depict the impacts of transmission and uations, as a result, they must be addressed using fractional differential equations with difference variables. Delay fractional differential equations (FDEs) are used to examine systems impacted by time delay and create essential mathematical models of real phenomena in engineering, mechanics, and economics [6],[7],[8],[9]. The idea of these systems' stability played an essential role in this context. It is worth mentioning that time delay in a control system might result in a closed-loop characteristic equation with sequential components. These sequential terms generally produced an unlimited number of isolated roots, making it difficult to analyze the stability of time-delay systems. Generally applicable algebraic solution for evaluating the stability of these systems .These processes need extensive knowledge and analysis[10]. It should be emphasized that the existence of a pure time delay in the control system, both in the regulating components and in the system state itself, might result in

undesirable reactions or even system instability .The Previous research has examined this subject in depth, concentrating on the application of the notion of Lyapunov's second principle and stability[11],[12].In addition to the matrix idea, understanding and analyzing the dynamic nature of the internet is essential in the field of computational science.

therefore, the predicating these electrons becomes an essential component of developing electrical models to regulate events .Furthermore, the system stability theories may be applied to a wide range of disciplines, like electrical engineering, mechanical engineering, biology, science, and environmental science. As a result, it is sufficient for the system to exist only from Lyapunov's point of view, but it must also be controllable, useful and desirable from a rational point of view. If a system is only stable within a narrow, unplanned range, it may be completely unstable. In this context, a comprehensive examination of different subgroups of the country's area is necessary. This study can provide more complete picture of the movement of the system in the given field. Furthermore, managing the system over limited time periods must be taken into account, as these frontier gains do not account for the system of paramount importance. As a result, multiple concepts of system stability have been proposed and explored in a series of earlier research. Researchers are always working to create new approaches and tools for understanding and ensuring the stability of complex systems.

As a result, these discoveries have been expanded [13] . In addition, new approaches for evaluating the stability of linear systems with a finite time horizon have been developed, with Amato and Colleagues

[14],[15]proposing the finite time horizon .The previous studies [16],[13],[17] have also addressed linear time delay analysis in the context of limited and practical stability. In addition, in the

context of finite-time stabilization and significant development with a focus on issues stabilization [18], theories of fractional dynamical systems are undergoing significant development .In a focus on issues such as robust and output-determined stability, internal stability, finite-time stability, practical stability, root location, strong controllability and observability, and other elements. Matignon [19],[20]researched the stability of finite-dimensional linear fractional differential systems in state space form (internal and exterior).

Stability is a fundamental concept in this field because it is concerned with the ability to ensure the stability of the system without the harmful influence of primitive angles or positions. In this context, researchers have previously presented models of dynamic systems and methods for analyzing their stability, such as "ultimate stability" and previous studies [21],[22] have also presented and discussed many features and results related to the robust stability of fractal systems, which include space systems with a continuous state. Uncertain. However, stability analysis of non-Lyapunov fractional systems remain a major challenges since algebraic tools cannot be used directly in this context due to the lack of the Roth-Horwitz criterion and fractional ordering.

A novel technique based on the Bellman-Gronwall method was recently introduced, as well as the development of the "classical" Bellman-Gronwall inequality. For a specific class of fractal systems, the emphasis was on the problem of sufficiency and the requirements that allow system paths to remain inside prescribed sets.

This paper will discuss and analyze these results, techniques, and recent advancements in the subject of fractal dynamic system stability. We will concentrate on systems with a short time delay and high stability, and we will try to make a difference.

Understanding time-stamped systems are critical for achieving effective applications in a wide range of sectors, including sophisticated robotics, intelligent medical systems, and industrial industries. The use of steps along with the generalized Gronwald inequality is a successful approach to creating dependable temporal stability criteria for complex systems, thanks to recent advances in control and theoretical engineering. Previous research, as indicated in sources[14],[15],[23],[24],[25],[26],[27] and [28] even though these sources have been widely referenced in earlier research The objective of this article is to highlight a new field of research and extend the concept of limited time stability of nonlinear partial-order delay systems by applying the step method together with the generalized Gronwald inequality.

Take into account the following Riemann–Hadamard and Riemann –Katugampola nonlinear differential control nonlocal system of fractional order

$$\begin{cases} {}^{RH}_{a}D^{\alpha}_{t}x(t) = Ax(t) + f\left(t, x(t), {}^{RK}_{a}D^{\alpha,\mu}_{t}x(t-s)\right) + Bu(t), & t \in [0.T] \\ x(t) = \vartheta(t), & t \in [-\tau, 0] \end{cases}$$

(1.1)

where ${}^{RH}_{a}D^{\alpha,\mu}_{t}$ denotes the Riemann–Hadamard derivative of order $\alpha > 0$ and $x(.) \in R^n$ for t∈ $[-\infty$,T] , and $0 < a < b < \infty$, ${}^{RK}_{a}D^{\alpha,\mu}_{t}$ denotes the Riemann −Katugampola Fractional derivative of order $0 < \alpha < 1$ and μ is a positive value .Also $f(.,.,.): [0.T] \times R^n \times R^n \to R^n$ and $B_{n\times m}$ is a control matrix and $u(t): [0, T] \to R^m$ is a control function. Finally the function $\vartheta(t)$ is a nonlocal function defined on $[-\tau, 0]$, $\tau \in (-\infty, 0)$.

The research strategy is as follows:The second section goes over the ${}^{RH}_{a}D^{\alpha}_{t}x(t)$calculus. In the third section, we looked at the theories of the existence and uniqueness of linear fractal differential equations. In Section 4, answer of the example using illustrations. Finally, in Section 5 , We state the conclusions of the article.

2. Preliminaries

Several definitions and lemmas related to fractional calculus are reviewed in this section.

Definition (2.1),[29]**:**

Let $\alpha > 0, \mu > 0$, $x \in L^1([a,b]), R)$, $0 < a < b < \infty$ .The left and right Riemann−Katugampola fractional derivatives (R-K FDs) of order α is defined by

$$ {}^{RK}_{a}D_t^{\alpha,\mu}x(t) = \frac{\mu^\alpha}{\Gamma(1-\alpha)}(t^{1-\mu}\frac{d}{dt})\int_a^t \frac{\tau^{\mu-1}}{(t^\mu - \tau^\mu)^\alpha}\, x(\tau)\ d\tau $$

$$ {}^{RK}_{b}D_t^{\alpha,\mu}x(t) = \frac{-\mu^\alpha}{\Gamma(1-\alpha)}(t^{1-\mu}\frac{d}{dt})\int_t^b \frac{\tau^{\mu-1}}{(\tau^\mu - t^\mu)^\alpha}\, x(\tau)\ d\tau. $$

Definition (2.2), [29]**:**

Let $\alpha > 0, \mu > 0$, $x \in L^1([a,b]), R)$, $0 < a < b < \infty$ .The left and right Riemann−Katugampola fractional integrals (R-KFIs) of order α is defined by

$$ {}^{RK}_{a}D_t^{-\alpha,\mu}x(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\mu - \tau^\mu}{\mu}\right)^{\alpha-1} x(\tau)\ \frac{d\tau}{\tau^{1-\mu}} $$

$$ {}^{RK}_{b}D_t^{-\alpha,\mu}x(t) = \frac{1}{\Gamma(\alpha)}\int_t^b \left(\frac{\tau^\mu - t^\mu}{\mu}\right)^{\alpha-1} x(\tau)\ \frac{d\tau}{\tau^{1-\mu}}. $$

Definition (2.3), [6]**:**

Let $\alpha \in (0,1), \mu > 0, [a,b] \in R, 0 < a < b < \infty$. The left and right Caputo −Katugampola fractional derivatives (C–KFDs) are defined respectively by

$$ {}^{CK}_{a}D_t^{\alpha,\mu}x(t) = \frac{\mu^\alpha}{\Gamma(1-\alpha)}(t^{1-\mu}\frac{d}{dt})\int_a^t \frac{\tau^{\mu-1}}{(t^\mu - \tau^\mu)^\alpha}\, (x(\tau) - x(a))\ d\tau, $$

$$ {}^{CK}_{b}D_t^{\alpha,\mu}x(t) = \frac{-\mu^\alpha}{\Gamma(1-\alpha)}(t^{1-\mu}\frac{d}{dt})\int_t^b \frac{\tau^{\mu-1}}{(\tau^\mu - t^\mu)^\alpha}\, (x(\tau) - x(b))\ d\tau. $$

Definition (2.4), [8]**:**

Let a, b be two real numbers with $0 < a < b$. Then the left and right Riemann–Hadamard fractional integrals (RH-FIs) of order $\alpha > 0$ for function $f: [a,b] \to R$ is defined by

$$ {}^{RH}_{a}D_t^{-\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \frac{x(\tau)}{\tau} d\tau \qquad \tau > a $$

$$ {}^{RH}_{b}D_t^{-\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_t^b \left(\log\frac{t}{\tau}\right)^{\alpha-1} \frac{x(\tau)}{\tau} d\tau \qquad \tau < b $$

Definition (2.5), [8]**:**

Let $\alpha > 0$ The set of real numbers with $n = [\alpha] + 1$,the left and right Riemann–Hadamard fractional derivatives (RH-FDs) of order $\alpha > 0$ defined by

$$ {}^{RH}_{a}D_t^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n \int_a^t \left(\log\frac{t}{\tau}\right)^{n-\alpha-1} \frac{x(\tau)}{\tau} d\tau \qquad \tau > a $$

$$ {}^{RH}_{b}D_t^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\left(-t\frac{d}{dt}\right)^n \int_t^b \left(\log\frac{t}{\tau}\right)^{n-\alpha-1} \frac{x(\tau)}{\tau} d\tau \qquad \tau < b $$

$$ {}^{RH}_{a}D_t^{n}x(t) = \left(t\frac{d}{dt}\right)^n x(t) \text{ and } {}^{RH}_{b}D_t^{n,\mu}x(t) = \left(-t\frac{d}{dt}\right)^n x(t) \text{ where } \alpha = n $$

Definition (2.6):

The Soboleve fractional normed space defined by $M^{\alpha,p}(\Omega) = \{u \in L^p(R)$and ${}^{RH}_{a}D_t^{\alpha}u(t)\ exist\ \}$ has a norm $\|x\|_{M^{\alpha,p}} = \|x\|_p + \|{}^{RH}_{a}D_t^{\alpha}x(t)\|_p$

Lemma (2.1), [9]**:**

Let $\alpha > 0, f(t) \geq 0$, be a nondecreasing function locally integrable on [0, T) (some $T \leq +\infty$) and $g(t) \geq 0$, nondecreasing continuous function defined on [0, T), $g(t) \leq M$ (constant), and suppose $x(t) \geq 0$ and locally integrable on [0, T) with

$$f_1(t) \leq f_2(t) + g(t)\,{}_a^R D_t^{-\alpha} f_1(t)$$

on this interval. Then $f_1(t) \leq f_2(t) E_\alpha(g(t)t^\alpha)$, $t \in [0,T)$, where E_α is the Mittag-Leffler function defined by $\boldsymbol{E_\alpha}(\varsigma) = \sum_{\mathcal{L}=0}^{\infty} \frac{\varsigma^{\mathcal{L}}}{\Gamma(\mathcal{L}\alpha+1)}$

Lemma (2.2),[12]**:**

Assume that $0 < f_1 \leq f_2, 0 < \alpha \leq 1$. Then

$$f_2^{\ \alpha} - f_1^{\ \alpha} \leq (f_2 - f_1)^\alpha$$

3.Problem Formulation for Proposal system

In this section, we look at the existence, uniqueness and stability theorems for -order nonlinear differential equations.

Theorem (3.1):

Let $\alpha > 0$, then the equality ${}_a^{RH}D_t^{\alpha,\mu}\left({}_a^{RH}D_t^{-\alpha,\mu} x\right) = x(t)$ is valid for any summable function $x(t) \in L_1(a,b), 0 < \alpha < 1$.

Proof:

By using definition (2.4),(2.5) we have

$${}_a^{RH}D_t^{\alpha}({}_a^{RH}D_t^{-\alpha} x) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\left(t\frac{d}{dt}\right)\int_a^t \left(\log\frac{t}{s}\right)^{-\alpha}\frac{1}{s}ds\left(\int_a^s\left(\log\frac{s}{\tau}\right)^{\alpha-1}\frac{x(\tau)}{\tau}d\tau\right)$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\left(t\frac{d}{dt}\right)\int_a^t \left(\log\frac{t}{s}\right)^{-\alpha}\frac{1}{s}ds\left(\int_a^s\left(\log\frac{s}{\tau}\right)^{\alpha-1}\frac{x(\tau)}{\tau}d\tau\right)$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\left(t\frac{d}{dt}\right)\int_a^t\left(\log\frac{t}{\tau}\right)\frac{x(\tau)}{\tau}d\tau = \frac{1}{\Gamma(1)}\left(t\frac{d}{dt}\right)\int_a^t\left(\log\frac{t}{\tau}\right)\frac{x(\tau)}{\tau}d\tau = x(t)$$

Lemma (3.2), [30]**:**

Let f be a continuous function on a rectangle $R = [a,b]\, x\, [c,d]$, then $\int\int f(x,y)dA$ have the $\int_a^b(\int_c^d f(x,y)dy)dx = \int_c^d(\int_a^b f(x,y)dx)dy$.

Let $\alpha > 0, \beta > 0, 1 \leq p \leq \infty, 0 < a < b < \infty$ and let $\mu \in R$ and $c \in R$ such that $\mu \geq c$. Then for $x \in X_c^p(a,b)$. Then ${}_a^{RH}D_t^{-\alpha}\left({}_a^{RH}D_t^{-\beta} x\right) = {}_a^{RH}D_t^{-\alpha-\beta} x$.

Proof:

By using Lemma (3.2), we have

Theorem (3.2):

${}_a^{RH}D_t^{-\alpha}\left({}_a^{RH}D_t^{-\beta} x(t)\right) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\int_a^\tau\left(\log\frac{\tau}{s}\right)^{\beta-1}\frac{x(s)}{s}ds\right)\frac{1}{\tau}d\tau$ .Now we have that,

$${}_a^{RH}D_t^{-\alpha}\left({}_a^{RH}D_t^{-\beta} x(t)\right) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\left(\int_\tau^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{\tau}{s}\right)^{\beta-1}\frac{x(s)}{s}ds\right)\frac{1}{\tau}d\tau$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\left(\int_\tau^t\left(\log\frac{t}{s}\right)^{\alpha+\beta-2}\frac{x(s)}{s}ds\right)\frac{1}{\tau}d\tau$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\left(\int_\tau^t\left(\log\frac{t}{s}\right)^{\alpha+\beta-2}\frac{x(s)}{s}ds\right)\frac{1}{\tau}d\tau$$

$$= \frac{1}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta)}\int_a^t\left(\left(\left(\log\left(\frac{t}{\tau}\right)\right)^{\alpha+\beta-1}x(\tau) - \left(\log\left(\frac{t}{t}\right)\right)^{\alpha+\beta-1}x(t)\right)\right)\frac{1}{\tau}d\tau$$

$$= \frac{1}{\Gamma(\alpha+\beta)}\int_a^t\left(\left(\left(\log\left(\frac{t}{\tau}\right)\right)^{\alpha+\beta-1}\frac{x(\tau)}{\tau}\right)\right)d\tau$$

$$= \frac{1}{\Gamma(\alpha+\beta)}\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha+\beta-1}\frac{x(\tau)}{\tau}dt$$

$${}_a^{RH}D_t^{-\alpha-\beta} x(t)$$

Lemma (3.3), [31]**:**

If $f(t) \in C^n[0,+\infty]$and $n-1<\alpha<n \in z^+$,

1. ${}^{RK}_{\ a}D_t^{\alpha,\mu}({}^{RK}_{\ a}D_t^{-\alpha,\mu}f(t)) = f(t)$

2. ${}^{CK}_{\ a}D_t^{\alpha,\mu}f(t) = {}^{RK}_{\ a}D_t^{\alpha,\mu}f(t) - \frac{\mu^\alpha f(a)}{\Gamma(1-\alpha)}(t^\mu - a^\mu)^{-\boldsymbol{\alpha}}$

Lemma (3.4):

The relation between Riemann–Hadamard fractional derivatives and Riemann–Hadamard fractional integral have the following formulation

$${}^{RH}_{\ a}D_t^{-\alpha}\left({}^{RH}_{\ a}D_t^{\alpha}f(t)\right) = \int_a^x \frac{d}{d\tau}\frac{f(\tau)}{\tau}\,d\tau - \sum_{k=1}^{n-1}\frac{1}{k!}\left(\log\frac{x}{a}\right)^k \frac{f^{[k]}(a)}{a}$$

for order $\alpha > 0, n = [\alpha]+1$, and for $n = 1$, we get the following relation

$${}^{RH}_{\ a}D_t^{-\alpha}\left({}^{RH}_{\ a}D_t^{\alpha}f(t)\right) = f(t) - f(a)$$

Proof:

Form Defintion (2.4), and definition (2.5), we have that

$${}^{RH}_{\ a}D_t^{-\alpha}\left({}^{RH}_{\ a}D_t^{\alpha,\mu}f(t)\right) = {}^{RH}_{\ a}D_t^{-\alpha}\left({}^{RH}_{\ a}D_t^{n-\alpha,\mu}f^{[n]}(t)\right) = {}^{RH}_{\ a}D_t^{n,\mu}f^{[n]}(t)$$

$$= \frac{1}{(n-1)!}\int_a^x \left(\log\frac{x}{\tau}\right)^{n-1} \frac{f^{[n]}(\tau)}{\tau}d\tau$$

$$= \frac{1}{(n-1)!}\int_a^x \left(\log\frac{x}{\tau}\right)^{n-1} \frac{d}{d\tau}\frac{f^{[n-1]}(\tau)}{\tau}d\tau$$

$$= \frac{1}{(n-2)!}\int_a^x \left(\log\frac{x}{\tau}\right)^{n-1} \frac{d}{d\tau}\frac{f^{[n-2]}(\tau)}{\tau}d\tau - \frac{1}{(n-1)!}\left(\log\frac{x}{a}\right)^{n-1}\frac{f^{[n-1]}(a)}{a}$$

$$= \frac{1}{(n-3)!}\int_a^x \left(\log\frac{x}{\tau}\right)^{n-1} \frac{d}{d\tau}\frac{f^{[n-3]}(\tau)}{\tau}d\tau - \sum_{k=n-2}^{n-1}\frac{1}{k!}\left(\log\frac{x}{a}\right)^k \frac{f^{[k]}(a)}{a}$$

$$= \cdots = \int_a^x \frac{d}{d\tau}\frac{f(\tau)}{\tau}\,d\tau - \sum_{k=1}^{n-1}\frac{1}{k!}\left(\log\frac{x}{a}\right)^k \frac{f^{[k]}(a)}{a}$$ for every $\alpha \in (0,1)$ then

$${}^{RH}_{\ a}D_t^{-\alpha}\left({}^{RH}_{\ a}D_t^{\alpha}f(t)\right) = f(t) - f(a)$$

Lemma (3.5):

A nonnegative function locally integrable a(t) on $0 \le t \le T$ (some $T \le +\infty$) and g(t) is a nonnegative, nondecreasing continuous function defined on $0 \le t \le T$ $g(t) \le M$ (constant), $\alpha > 0$, and suppose u(t) is nonnegative and locally integrable on $0 \le t \le T$ with

$$u(t) = a(t) + g(t)\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} u(s)\,ds$$

on this interval. Then

$$u(t) \le a(t) + \int_a^t \left[\sum_{n=1}^{\infty}\frac{\left(g(t)\Gamma(\alpha)\right)^n}{\Gamma(\alpha)\Gamma((k+1)\alpha)}\left(\log\frac{t}{s}\right)^{n\alpha-1} a(s)\right]ds$$

$$u(t) \le a(t)E_\alpha\left(g(t)\left(\log\frac{t}{\mathrm{a}}\right)^\alpha\right)$$

proof:

Let $B\varphi(t) = \frac{g(t)}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1}\varphi(s)\frac{-ds}{s}$, $t \ge 0$. Then $u(t) \le a(t) + Bu(t)$ imply

$$u(t) \le \sum_{k=0}^{n-1} B^k a(t) + B^n u(t)$$

Now

$$B^n u(t) \le \int_a^t \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}\left(\log\frac{t}{\tau}\right)^{n\alpha-1} a(\tau)\frac{d\tau}{\tau}$$

(3.2)

$$B^n u(t) \to 0 \ as\ n \longrightarrow +\infty \quad \forall t \in [0,T)$$

By induction the relation in (3.2) is true for $n = 1$. Assume that it is true for some $n = k$. now we need to prove for $n = k+1$, implies that

$$B^{k+1}u(t) = B(B^k) \leq \frac{g(t)}{\Gamma(\alpha)} \int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} \left[\int_a^s \frac{(g(s)\Gamma(\alpha))^k}{\Gamma(k\alpha)} \left(\log\frac{s}{\tau}\right)^{k\alpha-1} u(\tau)d\tau\right] \frac{-ds}{s}$$

Since $g(t)$ is nondecreasing, it follows that

$$B^{k+1}u(t) \leq \frac{(g(t))^{k+1}}{\Gamma(\alpha)} \int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} \left[\int_a^s \frac{(\Gamma(\alpha))^k}{\Gamma(k\alpha)} \left(\log\frac{s}{\tau}\right)^{k\alpha-1} u(\tau)d\tau\right] \frac{-ds}{s}$$

By interchanging the order of integration, we have

$$B^{k+1}u(t) \leq \frac{(g(t))^{k+1}}{\Gamma(\alpha)} \int_a^t \left[\int_\tau^t \frac{(\Gamma(\alpha))^k}{\Gamma(k\alpha)} \left(\log\frac{t}{s}\right)^{\alpha-1} \left(\log\frac{s}{\tau}\right)^{k\alpha-1} ds\right] u(\tau) \frac{-ds}{s}$$

$$B^{k+1}u(t) \leq \frac{(g(t))^{k+1}}{\Gamma(\alpha)} \int_a^t \left[\int_\tau^t \frac{(\Gamma(\alpha))^k}{\Gamma(k\alpha)} \left((1-z)\log\frac{t}{\tau}\right)^{\alpha-1} \left(z\log\frac{t}{\tau}\right)^{k\alpha-1} ds\right] u(\tau) \frac{-ds}{s}$$

$$B^{k+1}u(t) \leq \frac{(g(t))^{k+1}}{\Gamma(\alpha)} \int_a^t \left[\int_\tau^t \frac{(\Gamma(\alpha))^n}{\Gamma(n\alpha)} \left(\log\frac{t}{\tau}\right)^{\alpha-1} \left(\log\frac{t}{\tau}\right)^{k\alpha-1} (1-z)^\alpha z^{k\alpha-1}\ dz\right] u(\tau) \frac{-d\tau}{\tau^{1-\mu}}$$

$$B^{k+1}u(t) \leq \frac{(g(t))^{k+1}}{\Gamma(\alpha)} \int_a^t \left[\int_0^1 \frac{(\Gamma(\alpha))^k}{\Gamma(k\alpha)} \left(\log\frac{t}{\tau}\right)^{\alpha-1} \left(\log\frac{t}{\tau}\right)^{k\alpha-1} (1-z)^\alpha z^{k\alpha-1}\ dz\right] u(\tau) \frac{-d\tau}{\tau^{1-\mu}}$$

$$B^{k+1}u(t) \leq \frac{(g(t))^{k+1}}{\Gamma(\alpha)} \int_a^t \left[\frac{(\Gamma(\alpha))^k}{\Gamma(k\alpha)} \left(\log\frac{t}{\tau}\right)^{k\alpha+\alpha-1} \int_0^1 (1-z)^\alpha z^{k\alpha-1}\ dz\right] u(\tau) \frac{-d\tau}{\tau^{1-\mu}}$$

$$B^{k+1}u(t) \leq \frac{(g(t))^{k+1}}{\Gamma(\alpha)} \int_a^t \left[\frac{(\Gamma(\alpha))^k}{\Gamma(k\alpha)} \left(\log\frac{t}{s}\right)^{(k+1)\alpha-1} B(k\alpha,\alpha)\right] u(s) \frac{-ds}{s}$$

$$B^{k+1}u(t) \leq \frac{(g(t))^{k+1}}{\Gamma(\alpha)} \int_a^t \left[\frac{(\Gamma(\alpha))^k}{\Gamma(k\alpha)} \left(\log\frac{t}{s}\right)^{(k+1)\alpha-1} \frac{\Gamma(\alpha)\Gamma(n\alpha)}{\Gamma((k+1)\alpha)}\right] u(s) \frac{-ds}{s}$$

$$B^{k+1}u(t) \leq \int_a^t \left[\frac{(g(t)\Gamma(\alpha))^{k+1}}{\Gamma(\alpha)\Gamma((k+1)\alpha)} \left(\log\frac{t}{s}\right)^{(k+1)\alpha-1}\right] u(s) \frac{-ds}{s}$$

Since $B^n u(t) \leq \int_a^t \left[\frac{(g(t)\Gamma(\alpha))^n}{\Gamma(\alpha)\Gamma((k+1)\alpha)} \left(\log\frac{t}{s}\right)^{n\alpha-1}\right] u(s)ds \to 0, n \to \infty\ \forall t \in [0,T)$

$$u(t) \leq a(t) + \int_a^t \left[\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(\alpha)\Gamma(n\alpha)} \left(\log\frac{t}{s}\right)^{n\alpha-1} a(s)\right] \frac{-ds}{s} \qquad 0 \leq t \leq T$$

$$u(t) \leq a(t)\left(1 + \sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(\alpha)\Gamma(n\alpha)} \int_a^t \left[\left(\log\frac{t}{s}\right)^{n\alpha-1}\right] \frac{-ds}{s}\right)$$

$u(t) \leq a(t)\left(1 + \sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(\alpha)n\alpha\Gamma(n\alpha)} \left(\log\frac{t}{s}\right)^{n\alpha}\Big|_a^t\right)$, then

$u(t) \leq a(t)\left(1 + \sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(\alpha)n\alpha\Gamma(n\alpha)} \left(\log\frac{t}{a}\right)^{n\alpha}\right)$, we have that

$u(t) \leq a(t)\left(1 + \sum_{n=1}^{\infty} \frac{\left(g(t)\Gamma(\alpha)\left(\log\frac{t}{a}\right)\right)^n}{\Gamma(\alpha)n\alpha\Gamma(n\alpha)}\right)$, we obtain

$u(t) \leq a(t)\left(1 + \frac{1}{\Gamma(\alpha)} \sum_{n=1}^{\infty} \frac{\left(g(t)\Gamma(\alpha)\left(\log\frac{t}{a}\right)^\alpha\right)^n}{\Gamma(n\alpha+1)}\right)$, we get that

$u(t) \leq a(t)\left(\frac{1}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{\left(g(t)\Gamma(\alpha)\left(\log\frac{t}{a}\right)^\alpha\right)^n}{\Gamma(n\alpha+1)}\right)$, hence

$u(t) \leq a(t)E_\alpha\left(g(t)\left(\log\frac{t}{a}\right)^\alpha \Gamma(\alpha)\right)$, the lemma is complete.

Lemma (3.6):
let $\alpha > 0$,and $x(t) \in L^1([a,b]),R)$, $0 < a < b < \infty$ then $\|({}^{RH}_{a}D^{\alpha}_{t})x(t) - ({}^{RH}_{a}D^{\alpha}_{t})y(t)\|_{M^{\alpha,p}} \leq C_1(t)\|x(t)-y(t)\|_{M^{\alpha,p}}$ where $C_1(t) = \frac{\left(\log\frac{t}{a}\right)^{-\alpha}}{\Gamma(1-\alpha)}$
$\|({}^{RH}_{a}D^{\alpha}_{t})x(t) - ({}^{RH}_{a}D^{\alpha}_{t})y(t)\|_{M^{\alpha,p}} \leq M_1(t)(\|x(t)-y(t)\|_p$
where $M_1(t) = \left(\frac{\left(\log\frac{t}{a}\right)^{-\alpha}}{\Gamma(1-\alpha)} + \left(\frac{\left(\log\frac{t}{a}\right)^{-\alpha}}{\Gamma(1-\alpha)}\right)^2\right)$.

Proof.
From Definition(2.5), we have that

$$\left(({}^{RH}_{a}D^{\alpha}_{t})x(t) - ({}^{RH}_{a}D^{\alpha}_{t})y(t)\right) = \frac{1}{\Gamma(1-\alpha)}\left(t\frac{d}{dt}\right)\int_a^t\left(\log\frac{t}{\tau}\right)^{-\alpha}\frac{x(\tau)}{\tau}d\tau - \frac{1}{\Gamma(1-\alpha)}\left(t\frac{d}{dt}\right)\int_a^t\left(\log\frac{t}{\tau}\right)^{-\alpha}\frac{y(\tau)}{\tau}d\tau$$

$$= \frac{1}{\Gamma(1-\alpha)}\left(t\frac{d}{dt}\right)\int_a^t\left(\log\frac{t}{\tau}\right)^{-\alpha}\frac{(x(\tau)-y(\tau))}{\tau}d\tau$$

Therefore,

$$\|({}^{RH}_{a}D^{\alpha}_{t})x(t) - ({}^{RH}_{a}D^{\alpha}_{t})y(t)\|_{M^{\alpha,p}} = \frac{1}{\Gamma(1-\alpha)}\left(t\frac{d}{dt}\right)\int_a^t\left(\log\frac{t}{\tau}\right)^{-\alpha}\frac{\|x(\tau)-y(\tau)\|_{M^{\alpha,p}}}{\tau}d\tau$$

$$\|({}^{RH}_{a}D^{\alpha}_{t})x(t) - ({}^{RH}_{a}D^{\alpha}_{t})y(t)\|_{M^{\alpha,p}} = \frac{1}{\Gamma(1-\alpha)}\|x(\tau)-y(\tau)\|_{M^{\alpha,p}}\left(t\frac{d}{dt}\right)\int_a^t\left(\log\frac{t}{\tau}\right)^{-\alpha}\frac{d\tau}{\tau}$$

$$\|({}^{RH}_{a}D^{\alpha}_{t})x(t) - ({}^{RH}_{a}D^{\alpha}_{t})y(t)\|_{M^{\alpha,p}} = \frac{1}{\Gamma(1-\alpha)}\|x(\tau)-y(\tau)\|_{M^{\alpha,p}}\left(t\frac{d}{dt}\right)\int_a^t\left(\log\frac{t}{\tau}\right)^{-\alpha}\frac{d\tau}{\tau}$$

$$\|({}^{RH}_{a}D^{\alpha}_{t})x(t) - ({}^{RH}_{a}D^{\alpha}_{t})y(t)\|_{M^{\alpha,p}} = \frac{1}{(1-\alpha)\Gamma(1-\alpha)}\|x(\tau)-y(\tau)\|_{M^{\alpha,p}}\left(t\frac{d}{dt}\right)\left(\log\frac{t}{\tau}\right)^{1-\alpha}\Big|_a^t$$

$$\|({}^{RH}_{a}D^{\alpha}_{t})x(t) - ({}^{RH}_{a}D^{\alpha}_{t})y(t)\|_{M^{\alpha,p}} = \frac{1}{(1-\alpha)\Gamma(1-\alpha)}\|x(\tau)-y(\tau)\|_{M^{\alpha,p}}\left(t\frac{d}{dt}\right)\left(\log\frac{t}{a}\right)^{1-\alpha}$$

$$\|({}^{RH}_{a}D^{\alpha}_{t})x(t) - ({}^{RH}_{a}D^{\alpha}_{t})y(t)\|_{M^{\alpha,p}} = \frac{-1}{\Gamma(1-\alpha)}\|x(\tau)-y(\tau)\|_{M^{\alpha,p}}(t)\left(\log\frac{t}{a}\right)^{-\alpha}$$

$$\|({}^{RH}_{a}D^{\alpha}_{t})x(t) - ({}^{RH}_{a}D^{\alpha}_{t})y(t)\|_{M^{\alpha,p}} = \frac{\left(\log\frac{t}{a}\right)^{-\alpha}}{\Gamma(1-\alpha)}\|x(\tau)-y(\tau)\|_{M^{\alpha,p}}$$

set $C_1(t) = \frac{\left(\log\frac{t}{a}\right)^{-\alpha}}{\Gamma(1-\alpha)}$, we get that

$$\|({}^{RH}_{a}D^{\alpha}_{t})x(t) - ({}^{RH}_{a}D^{\alpha}_{t})y(t)\|_{M^{\alpha,p}} \leq C_1(t)\|x(t)-y(t)\|_{M^{\alpha,p}}$$

$$\leq C_1(t)(\|x(t)-y(t)\|_p + \left\|{}^{RH}_{a}D^{\alpha,\mu}_{t}(x(t)-y(t))\right\|_p)$$

$$\leq C_1(t)(\|x(t)-y(t)\|_p + \left\|\frac{1}{\Gamma(1-\alpha)}\left(t\frac{d}{dt}\right)\int_a^t\left(\log\frac{t}{\tau}\right)^{-\alpha}\frac{x(\tau)-y(\tau)}{\tau}d\tau\right\|_p)$$

$$\leq C_1(t)(\|x(t)-y(t)\|_p + \|x(t)-y(t)\|_p\frac{1}{\Gamma(1-\alpha)}\left(t\frac{d}{dt}\right)\int_a^t\left(\log\frac{t}{\tau}\right)^{-\alpha}\frac{d\tau}{\tau})$$

$$\leq C_1(t)\left(\|x(t)-y(t)\|_p + \|x(t)-y(t)\|_p\frac{-\left(\log\frac{t}{a}\right)^{-\alpha}}{\Gamma(1-\alpha)}\right)$$

(3.3)

$$\leq \left(\|x(t)-y(t)\|_p\left(\frac{\left(\log\frac{t}{a}\right)^{-\alpha}}{\Gamma(1-\alpha)} + C_1(t)\frac{\left(\log\frac{t}{a}\right)^{-\alpha}}{\Gamma(1-\alpha)}\right)\right)$$

$\leq (\|x(t)-y(t)\|_p \left(\frac{\left(\log\frac{t}{a}\right)^{-\alpha}}{\Gamma(1-\alpha)} + \left(\frac{\left(\log\frac{t}{a}\right)^{-\alpha}}{\Gamma(1-\alpha)}\right)^2\right))$

Set $M_1(t) = \left(\left(\frac{\left(\log\frac{t}{a}\right)^{-\alpha}}{\Gamma(1-\alpha)} + \left(\frac{\left(\log\frac{t}{a}\right)^{-\alpha}}{\Gamma(1-\alpha)}\right)^2\right)$, hence

$\|({}^{RH}_{a}D^{\alpha}_{t})x(t) - ({}^{RH}_{a}D^{\alpha}_{t})y(t)\|_{M^{\alpha,p}} \leq M_1(t)(\|x(t)-y(t)\|_p$

(3.4)

Lemma (3.7):

Let $\alpha > 0$, for $x(t) \in L^1([a,b]),R)$, $0 < a < b < \infty$

Then

1. $\|{}^{RH}_{a}D^{-\alpha}_{t}x(t) - {}^{RH}_{a}D^{-\alpha}_{t}y(t)\|_{M^{\alpha,p}} \leq C_2(t)\|x(t)-y(t)\|_{M^{\alpha,p}}$

where $C_2(t) = \frac{-\left(\log\frac{t}{a}\right)^{\alpha}}{\alpha\Gamma(\alpha)}$.

2. $\|{}^{RH}_{a}D^{-\alpha}_{t}x(t) - {}^{RH}_{a}D^{-\alpha}_{t}y(t)\|_{M^{\alpha,p}} \leq M_2(t)\big(\|x(t)-y(t)\|_p\big)$ where $M_2(t) = \left(\frac{\left(lo\,g\frac{t}{a}\right)^{\alpha}}{\alpha\Gamma(\alpha)} + \frac{1}{\alpha\Gamma(\alpha)\Gamma(1-\alpha)}\right)$.

Proof:

1. ${}^{RH}_{a}D^{-\alpha}_{t}x(t) - {}^{RH}_{a}D^{-\alpha}_{t}y(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \frac{x(\tau)}{\tau}d\tau - \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \frac{y(\tau)}{\tau}d\tau$

$${}^{RH}_{a}D^{-\alpha}_{t}x(t) - {}^{RH}_{a}D^{-\alpha}_{t}y(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \frac{x(\tau)-y(\tau)}{\tau}d\tau$$

Therefore

$$\|{}^{RH}_{a}D^{-\alpha}_{t}x(t) - {}^{RH}_{a}D^{-\alpha}_{t}y(t)\|_{M^{\alpha,p}} \leq \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \frac{\|x(t)-y(t)\|_{M^{\alpha,p}}}{\tau}d\tau$$

$$\|{}^{RH}_{a}D^{-\alpha}_{t}x(t) - {}^{RH}_{a}D^{-\alpha}_{t}y(t)\|_{M^{\alpha,p}} \leq \frac{\|x(t)-y(t)\|_{M^{\alpha,p}}}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \frac{d\tau}{\tau}$$

$$\|{}^{RH}_{a}D^{-\alpha}_{t}x(t) - {}^{RH}_{a}D^{-\alpha}_{t}y(t)\|_{M^{\alpha,p}} \leq \frac{\left(\log\frac{t}{a}\right)^{\alpha}}{\alpha\Gamma(\alpha)}\|x(t)-y(t)\|_{M^{\alpha,p}}$$

set $C_2(t) = \frac{\left(\log\frac{t}{a}\right)^{\alpha}}{\alpha\Gamma(\alpha)}$ we get

2. $\|{}^{RH}_{a}D^{-\alpha}_{t}x(t) - {}^{RH}_{a}D^{-\alpha}_{t}y(t)\|_{M^{\alpha,p}} \leq C_2(t)\|x(t)-y(t)\|_{M^{\alpha,p}}$

(3.5)

$$\|{}^{RH}_{a}D^{-\alpha}_{t}x(t) - {}^{RH}_{a}D^{-\alpha}_{t}y(t)\|_{M^{\alpha,p}} \leq C_2(t)\left(\|x(t)-y(t)\|_p\left(1 - \frac{\left(\log\frac{t}{a}\right)^{-\alpha}}{\Gamma(1-\alpha)}\right)\right)$$

$$\|{}^{RH}_{a}D^{-\alpha}_{t}x(t) - {}^{RH}_{a}D^{-\alpha}_{t}y(t)\|_{M^{\alpha,p}} \leq \left(\|x(t)-y(t)\|_p\left(\frac{\left(lo\,g\frac{t}{a}\right)^{\alpha}}{\alpha\Gamma(\alpha)} + \frac{\left(lo\,g\frac{t}{a}\right)^{\alpha}}{\alpha\Gamma(\alpha)}\frac{\left(\log\frac{t}{a}\right)^{-\alpha}}{\Gamma(1-\alpha)}\right)\right)$$

Set $M_2(t) = \left(\frac{\left(lo\,g\frac{t}{a}\right)^{\alpha}}{\alpha\Gamma(\alpha)} + \frac{1}{\alpha\Gamma(\alpha)\Gamma(1-\alpha)}\right)$.

(3.6)

Then, $\|{}^{RH}_{a}D^{-\alpha}_{t}x(t) - {}^{RH}_{a}D^{-\alpha}_{t}y(t)\|_{M^{\alpha,p}} \leq M_2(t)\big(\|x(t)-y(t)\|_p\big)$

Lemma (3.8):

Let $\alpha > 0, \mu > 0$,and $x(t) \in L^1([a,b]),R)$, $0 < a < b < \infty$ then $\left\| \begin{matrix} ({}^{RK}_{a}D_t^{\alpha,\mu})x(t) \; - \\ ({}^{RK}_{a}D_t^{\alpha,\mu})y(t)) \end{matrix} \right\|_{M^{\alpha,p}} \leq$ $C_3(t)\|x(\tau) - y(\tau)\|$ when $C_3(t) = \frac{\mu^\alpha}{\Gamma(1-\alpha)}\left((t^\mu - a^\mu)\right)$.

Proof:

$\left(({}^{RK}_{a}D_t^{\alpha,\mu})x(t) - ({}^{RK}_{a}D_t^{\alpha,\mu})y(t)\right) = \frac{\mu^\alpha}{\Gamma(1-\alpha)}\left(t^{(1-\mu)}\frac{d}{dt}\right)\int_0^t \frac{\tau^{(\mu-1)}}{(t^\mu-\tau^\mu)^\alpha}x(\tau)d\tau -$ $\frac{\mu^\alpha}{\Gamma(1-\alpha)}\left(t^{(1-\mu)}\frac{d}{dt}\right)\int_a^t \frac{\tau^{(\mu-1)}}{(t^\mu-\tau^\mu)^\alpha}y(\tau)d\tau$

$$= \frac{\mu^\alpha}{\Gamma(1-\alpha)}\left(t^{1-\mu}\frac{d}{dt}\right)\left[\int_a^t \frac{\tau^{(\mu-1)}}{(t^\mu-\tau^\mu)^\alpha}\left(x(\tau)-y(\tau)\right)d\tau\right]$$

Therefore

$$\left\|({}^{RK}_{a}D_t^{\alpha,\mu})x(t) - ({}^{RK}_{a}D_t^{\alpha,\mu})y(t)\right\|_{M^{\alpha,p}} \leq \frac{\mu^\alpha}{\Gamma(1-\alpha)}\left(t^{1-\mu}\frac{d}{dt}\right)\left[\int_a^t \frac{\tau^{(\mu-1)}}{(t^\mu-\tau^\mu)^\alpha}\|x(\tau)-y(\tau)\|_{M^{\alpha,p}}\,d\tau\right]$$

$$\left\|({}^{RK}_{a}D_t^{\alpha,\mu})x(t) - ({}^{RK}_{a}D_t^{\alpha,\mu})y(t)\right\|_{M^{\alpha,p}} \leq \|x(t)-y(t)\|_{M^{\alpha,p}}\frac{\mu^\alpha}{\Gamma(1-\alpha)}\left(t^{1-\mu}\frac{d}{dt}\right)\left[\int_a^t \frac{\tau^{(\mu-1)}}{(t^\mu-\tau^\mu)^\alpha}d\tau\right]$$

$$\left\|({}^{RK}_{a}D_t^{\alpha,\mu})x(t) - ({}^{RK}_{a}D_t^{\alpha,\mu})y(t)\right\|_{M^{\alpha,p}} \leq \|x(t)-y(t)\|_{M^{\alpha,p}}\frac{\mu^\alpha}{\mu\Gamma(1-\alpha)}\left(t^{1-\mu}\frac{d}{dt}\right)\left[\int_a^t -(t^\mu-\tau^\mu)^{-\alpha}\mu\tau^{(\mu-1)}d\tau\right]$$

$$\left\|({}^{RK}_{a}D_t^{\alpha,\mu})x(t) - ({}^{RK}_{a}D_t^{\alpha,\mu})y(t)\right\|_{M^{\alpha,p}} \leq \|x(t)-y(t)\|_{M^{\alpha,p}}\frac{\mu^\alpha}{\mu\Gamma(1-\alpha)}\left(t^{1-\mu}\frac{d}{dt}\right)\left[\frac{(t^\mu-\tau^\mu)^{-\alpha+1}}{-\alpha+1}\Big|_a^t\right]$$

$$\left\|({}^{RK}_{a}D_t^{\alpha,\mu})x(t) - ({}^{RK}_{a}D_t^{\alpha,\mu})y(t)\right\|_{M^{\alpha,p}} \leq \|x(t)-y(t)\|_{M^{\alpha,p}}\frac{\mu^\alpha}{\mu\Gamma(1-\alpha)}\left(t^{1-\mu}\frac{d}{dt}\right)\left[\frac{0-(t^\mu-a^\mu)^{-\alpha+1}}{-\alpha+1}\right]$$

$$\left\|({}^{RK}_{a}D_t^{\alpha,\mu})x(t) - ({}^{RK}_{a}D_t^{\alpha,\mu})y(t)\right\|_{M^{\alpha,p}} \leq \|x(t)-y(t)\|_{M^{\alpha,p}}\frac{\mu^\alpha}{\mu\Gamma(1-\alpha)}\left(t^{1-\mu}\frac{d}{dt}\right)\left[\frac{(t^\mu-a^\mu)^{-\alpha+1}}{-\alpha+1}\right]$$

$$\left\|({}^{RK}_{a}D_t^{\alpha,\mu})x(t) - ({}^{RK}_{a}D_t^{\alpha,\mu})y(t)\right\|_{M^{\alpha,p}} \leq \|x(t)-y(t)\|_{M^{\alpha,p}}\frac{\mu^\alpha}{\mu\Gamma(1-\alpha)}\left(t^{1-\mu}\frac{d}{dt}\right)\left[\frac{(t^\mu-a^\mu)^{-\alpha+1}}{-\alpha+1}\right]$$

$$\left\|({}^{RK}_{a}D_t^{\alpha,\mu})x(t) - ({}^{RK}_{a}D_t^{\alpha,\mu})y(t)\right\|_{M^{\alpha,p}} \leq \|x(t)-y(t)\|_{M^{\alpha,p}}\frac{\mu^\alpha}{\mu\Gamma(1-\alpha)}\left(t^{1-\mu}\frac{(-\alpha+1)(t^\mu-a^\mu)^{-\alpha}\mu t^{\mu-1}}{-\alpha+1}\right)$$

$$\left\|({}^{RK}_{a}D_t^{\alpha,\mu})x(t) - ({}^{RK}_{a}D_t^{\alpha,\mu})y(t)\right\|_{M^{\alpha,p}} \leq \frac{\mu^\alpha}{\Gamma(1-\alpha)}\left((t^\mu-a^\mu)^{-\alpha}\right)\|x(t)-y(t)\|_{M^{\alpha,p}}$$
(3.7)

set $C_3(t) = \frac{\mu^\alpha}{\Gamma(1-\alpha)}(t^\mu - a^\mu)^{-\alpha}$, we get

$\left\|({}^{RK}_{a}D_t^{\alpha,\mu})x(t) - ({}^{RK}_{a}D_t^{\alpha,\mu})y(t)\right\|_{M^{\alpha,p}} \leq C_3(t)(\|x(t)-y(t)\|_{M^{\alpha,p}})$,now we get

$$\left\|({}^{RK}_{a}D_t^{\alpha,\mu})x(t) - ({}^{RK}_{a}D_t^{\alpha,\mu})y(t)\right\|_{M^{\alpha,p}} \leq C_3(t)\left(\|x(t)-y(t)\|_p + \left\|{}^{RH}_{a}D_t^{\alpha,\mu}\left(x(t)-y(t)\right)\right\|_p\right)$$

$$\leq C_3(t)(\|x(t)-y(t)\|_p + \left\|\frac{1}{\Gamma(1-\alpha)}\left(t\frac{d}{dt}\right)\int_a^t\left(\log\frac{t}{\tau}\right)^{-\alpha}\frac{x(\tau)-y(\tau)}{\tau}d\tau\right\|_p)$$

$$\leq C_3(t)(\|x(t)-y(t)\|_p + \|x(t)-y(t)\|_p\frac{1}{\Gamma(1-\alpha)}\left(t\frac{d}{dt}\right)\int_a^t\left(\log\frac{t}{\tau}\right)^{-\alpha}\frac{d\tau}{\tau})$$

$$\leq C_3(t)\left(\|x(t)-y(t)\|_p+\|x(t)-y(t)\|_p\frac{\left(\log\frac{t}{a}\right)^{-\alpha}}{\Gamma(1-\alpha)}\right)$$

$$\leq (\|x(t)-y(t)\|_p\left(\frac{\mu^{\alpha}(t^{\mu}-a^{\mu})^{-\alpha}}{\Gamma(1-\alpha)}+C_3(t)\frac{\left(\log\frac{t}{a}\right)^{-\alpha}}{\Gamma(1-\alpha)}\right))$$

$$\leq (\|x(t)-y(t)\|_p\left(\frac{\mu^{\alpha}(t^{\mu}-a^{\mu})^{-\alpha}}{\Gamma(1-\alpha)}+\left(\frac{\mu^{\alpha}(t^{\mu}-a^{\mu})^{-\alpha}}{\Gamma(1-\alpha)}\right)\frac{\left(\log\frac{t}{a}\right)^{-\alpha}}{\Gamma(1-\alpha)}\right)$$

(3.8)t

$M_3(t)=\left(\frac{\mu^{\alpha}(t^{\mu}-a^{\mu})^{-\alpha}}{\Gamma(1-\alpha)}+\left(\frac{\mu^{\alpha}(t^{\mu}-a^{\mu})^{-\alpha}}{\Gamma(1-\alpha)}\right)\frac{\left(\log\frac{t}{a}\right)^{-\alpha}}{\Gamma(1-\alpha)}\right)$, hence

$$\|({}^{RH}_{a}D^{\alpha}_{t})x(t)\ -\ ({}^{RH}_{a}D^{\alpha}_{t})y(t)\|_{M^{\alpha,p}}\leq M_3(t)(\|x(t)-y(t)\|_p$$

Theorem (3.3):
Let $x:[-\tau,T]\to R^m$ Be a continuous differential function , $\tau>0$ then x(t) is a solution of the Riemann–Hadamard Fractional order nonlinear differential control nonlocal system (1.1),if and only if

$$x(t)=\begin{cases}{}^{RH}_{a}D^{-\alpha}_{t}\left(Ax(t)+f\left(t,x(t),{}^{RK}_{a}D^{\alpha,\mu}_{t}x(t-s)\right)+Bu(t)\right)+x(a)\\ \\ \phi(t) \qquad for-\tau\leq t\leq 0\end{cases}$$

(3.9)

Proof:
For $-\tau\ \leq\ t\ \leq\ 0$, we have the solution is $x(t)=\ \phi(t)$, now from lemma (3.4) , we have that

$x(t)={}^{RH}_{a}D^{-\alpha}_{t}\left(Ax(t)+f\left(t,x(t),{}^{RK}_{a}D^{\alpha,\mu}_{t}x(t-s)\right)+Bu(t)\right)+x(a)\quad 0\ \leq\ t\ \leq\ T$,
implies that

$x(t)-x(a)=\left(Ax(t)+f\left(t,x(t),{}^{RK}_{a}D^{\alpha,\mu}_{t}x(t-s)\right)+Bu(t)\right)$

, ${}^{RH}_{a}D^{-\alpha}_{t}({}^{RH}_{a}D^{\alpha}_{t}x(t))\ ={}^{RK}_{a}D^{-\alpha,\mu}_{t}\left(Ax(t)+f\left(t,x(t),{}^{RK}_{a}D^{\alpha,\mu}_{t}x(t-s)\right)+Bu(t)\right)$ By using lemma (3.4), we obtain

${}^{RH}_{a}D^{\alpha}_{t}x(t)=Ax(t)+f\left(t,x(t),{}^{RK}_{a}D^{\alpha,\mu}_{t}x(t-s)\right)+Bu(t).$

The other side of proving which given by:

${}^{RH}_{a}D^{\alpha}_{t}x(t)=Ax(t)+f\left(t,x(t),{}^{RK}_{a}D^{\alpha,\mu}_{t}x(t-s)\right)+Bu(t)$

${}^{RH}_{a}D^{-\alpha}_{t}({}^{RH}_{a}D^{\alpha}_{t}x(t))={}^{RH}_{a}D^{-\alpha}_{t}\left(Ax(t)+f\left(t,x(t),{}^{RK}_{a}D^{\alpha,\mu}_{t}x(t-s)\right)+Bu(t)\right)$

$x(t)-x(a)={}^{RH}_{a}D^{-\alpha}_{t}\left(Ax(t)+f\left(t,x(t),{}^{RK}_{a}D^{\alpha,\mu}_{t}x(t-s)\right)+Bu(t)\right)$, therefore

$x(t)={}^{RH}_{a}D^{-\alpha}_{t}\left(Ax(t)+f\left(t,x(t),{}^{RK}_{a}D^{\alpha,\mu}_{t}x(t-s)\right)+Bu(t)\right)+x(a),$

is the solution of (1.1).

Theorem (3.4):
Consider the Riemann–Hadamard and Riemann −Katugampola Fractional order nonlinear differential control nonlocal system (1.1) with $(f(t,0,0)=[0,\dots,0]^T)$.Then has a unique continuous solution.

proof:
Let $x_1\ (t)$ and $x_2(t)$ be any two different solutions to system (1), then $x(t)$ and $y(t)$ both satisfy the formulation of solution (1.1). Also, let $\xi(t)\ =x_1(t)-x_2(t)=\phi(t)-\phi(t)=0$, one can obtain $\xi(t)\ =\ 0\ for\ -\tau\ \leq\ t\ \leq\ 0$. Hence the system

(1.1) has a unique continuous solution for $-\tau \le t \le 0$.
Now, for $0 \le t \le T$, we have that

$$\xi(t) = x_1(t)\text{-}x_2(t) = {}^{RH}_{a}D_t^{-\alpha}\left(Ax(t) + f\left(t, x(t), {}^{RK}_{a}D_t^{\alpha,\mu}x(t-s)\right) + Bu(t)\right) -$$

$${}^{RH}_{a}D_t^{-\alpha}\left(Ay(t) + f\left(t, y(t), {}^{RK}_{a}D_t^{\alpha,\mu}y(t-s)\right) + Bu(t)\right)$$

$$\xi(t) = {}^{RK}_{a}D_t^{-\alpha,\mu}\left(A\xi(t) + f\left(t, \xi(t), {}^{RK}_{a}D_t^{\alpha,\mu}\xi(t-s)\right)\right) + \xi(a)$$

$$\xi(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(A\xi(\tau) + f\left(\tau, \xi(\tau), {}^{RK}_{a}D_t^{\alpha,\mu}\xi(\tau-s)\right) - f(\tau,0,0)\right)\frac{d\tau}{\tau} + \xi(a)$$

Where $0 \le t \le \tau$, we get that

$$\xi(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(A\xi(\tau) + f\left(t, \xi(\tau), {}^{RK}_{a}D_t^{\alpha,\mu}\xi(\tau-s)\right) - f(\tau,0,0)\right)\frac{d\tau}{\tau} + \xi(a)$$

$$\|\xi(t)\|_{M^{\alpha,p}} = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\left\|A\xi(\tau) + f\left(\tau, \xi(\tau), {}^{RK}_{a}D_t^{\alpha,\mu}\xi(\tau-s)\right) - f(\tau,0,0)\right\|_{M^{\alpha,p}}\frac{d\tau}{\tau} + \|\xi(a)\|_{M^{\alpha,p}}$$

$$\|\xi(t)\|_{M^{\alpha,p}} = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\left\|A\xi(\tau) + L\left(t, \xi(\tau), {}^{RK}_{a}D_t^{\alpha,\mu}\xi(\tau-s)\right)\right\|_{M^{\alpha,p}}\frac{d\tau}{\tau} + \|\xi(a)\|_{M^{\alpha,p}}$$

$$\|\xi(t)\|_{M^{\alpha,p}} = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\left\|A\xi(\tau) + L\big(\xi(\tau) + C_3(\tau)\xi(\tau-s)\big)\right\|_{M^{\alpha,p}}\frac{d\tau}{\tau} + \|\xi(a)\|_{M^{\alpha,p}}$$

$$\|\xi^*(t)\|_{M^{\alpha,p}} = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\left\|A\xi^*(\tau) + L\big(\xi^*(\tau) + C_3(t)\xi^*(\tau)\big)\right\|_{M^{\alpha,p}}\frac{d\tau}{\tau} + \|\xi(a)\|_{M^{\alpha,p}}$$

Where $\xi^*(t) = \sup_{\vartheta\in[-\tau,0]}\|\xi(t+\vartheta)\|$

$$\|\xi^*(\tau)\|_{M^{\alpha,p}} \le \|A + L(1 + C_3(t))\|_{M^{\alpha,p}}\frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\|\xi^*(\tau)\|_{M^{\alpha,p}}\frac{d\tau}{\tau}$$

Set g(t)= $(\|A + L(1 + C_3(t))\|_{M^{\alpha,p}})$

$$\|\xi^*(\tau)\|_{M^{\alpha,p}} \le \mathrm{g}(t)\frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\|\xi^*(\tau)\|_{M^{\alpha,p}}\frac{d\tau}{\tau}$$

$$\|\xi^*(t)\| \le 0E_\alpha\left(\mathrm{g}(t)\left(\log\frac{t}{a}\right)^{\alpha}\Gamma(\alpha)\right)$$

$$\|\xi^*(\tau)\|_{M^{\alpha,p}} \le 0$$

$$\|\xi^*(\tau)\|_{M^{\alpha,p}} \le x(t) - y(t) \le 0$$

$$x(t) = y(t)$$

If $\tau \le t < T$ *then* $\xi(t) = x(t) - y(t)$

$$\|\xi^*(\tau)\|_{M^{\alpha,p}} \le \frac{(\|A + L(1 + C_3(t))\|_{M^{\alpha,p}})}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\|\xi^*(\tau)\|_{M^{\alpha,p}}\frac{d\tau}{\tau^{1-\mu}}$$

when $\hat{L}_1 = (\|A + L(1 + C_3(t))\|_{M^{\alpha,p}})$

$$\|\xi^*(\tau)\|_{M^{\alpha,p}} \le (\|A + L(1 + C_3(t))\|_{M^{\alpha,p}})\frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\|\xi^*(\tau)\|_{M^{\alpha,p}}\frac{d\tau}{\tau^{1-\mu}}$$

$$\|\xi^*(\tau)\|_{M^{\alpha,p}} \le \hat{L}_1\frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\|\xi^*(\tau)\|_{M^{\alpha,p}}\frac{d\tau}{\tau^{1-\mu}}$$

$$\|\xi^*(\tau)\|_{M^{\alpha,p}} \le \hat{L}_1\frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\|\xi^*(\tau)\|_{M^{\alpha,p}}\frac{d\tau}{\tau^{1-\mu}}$$

$$\|\xi^*(\tau)\|_{M^{\alpha,p}} \le (\hat{L}_1)\frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\|\xi^*(\tau)\|_{M^{\alpha,p}}\frac{d\tau}{\tau^{1-\mu}}$$

$\|\xi^*(\tau)\|_{M^{\alpha,p}} \leq 0E_\alpha(\hat{L}_1)\left(\left(\log\frac{t}{a}\right)^\alpha\right)$

$\|\xi^*(\tau)\|_{M^{\alpha,p}} \leq 0$,Then $x(t) - y(t) = 0$

Then system (1.1) has a unique continuous solution.

3.1 Stability of the Riemann–Hadamard and Riemann −Katugampola Fractional order nonlinear differential control nonlocal system with maximal interval (0,T]

Theorem (3.1.5):

The solution to the Riemann-Hadamard and Riemann-Katugampola −Katugampola fractional order nonlinear differential control nonlocal system (1.1) is in the equation (3.9). Then the following inequalities hold:

$$\| x(t) \| \leq \delta(t)E_\alpha\left(\sigma(t)\left(\log\frac{t}{\mathrm{a}}\right)^\alpha\right)$$

(3.10)

Proof:

Since $x(t)$ has the following formulation,

$$x(t) = \begin{cases} {}^{RH}_{a}D_t^{-\alpha,\mu}\left(Ax(t) + f\left(t, x(t), {}^{RK}_{a}D_t^{\alpha,\mu}x(t-s)\right) + Bu(t)\right) + x(a) & \\ \vartheta(t) & t \in [-\tau, 0] \end{cases}$$

. Thus, for $0 \leq t \leq \mathrm{T}$, we have

$$x(t) = \frac{1}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(Ax(t) + f\left(t, x(t), {}^{RK}_{a}D_t^{\alpha,\mu}x(t-s)\right) + Bu(t)\right)\frac{d\tau}{\tau} + \|x(a)\|_{M^{\alpha,p}}$$

$$\|x(t)\|_{M^{\alpha,p}} = \frac{1}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left\|Ax(t) + f\left(t, x(t), {}^{RK}_{a}D_t^{\alpha,\mu}x(t-s)\right) + Bu(t)\right\|_{M^{\alpha,p}}\frac{d\tau}{\tau} + \|x(a)\|_{M^{\alpha,p}}$$

$$\|x(t)\|_{M^{\alpha,p}} = \left\|Ax(t) + f\left(t, x(t), {}^{RK}_{a}D_t^{\alpha,\mu}x(t-s)\right) - f(t,0,0) + Bu(t)\right\|_{M^{\alpha,p}}\frac{1}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{d\tau}{\tau} + \|x(a)\|_{M^{\alpha,p}}$$

$$\|x(t)\|_{M^{\alpha,p}} = \left\|Ax(t) + L\left(x(t) + {}^{RK}_{a}D_t^{\alpha,\mu}x(t-s)\right) + Bu(t)\right\|_{M^{\alpha,p}}\frac{1}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{d\tau}{\tau} + \|x(a)\|_{M^{\alpha,p}}$$

$$\|x(t)\|_{M^{\alpha,p}} = \left\|Ax(t) + L\left(x(t) + {}^{RK}_{a}D_t^{\alpha,\mu}x(t-s)\right) + Bu(t)\right\|_{M^{\alpha,p}}\frac{1}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{d\tau}{\tau} + \|x(a)\|_{M^{\alpha,p}}$$

$$\|x(t)\|_{M^{\alpha,p}} \leq \left(\|Ax(t)\|_{M^{\alpha,p}} + \|L(x(t))\|_{M^{\alpha,p}} + \|L(C_3(t)x(t-s))\|_{M^{\alpha,p}}\right)\frac{1}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{d\tau}{\tau} + \|Bu(t)\|_{M^{\alpha,p}}\frac{1}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{d\tau}{\tau} + \|x(a)\|_{M^{\alpha,p}}$$

$$\|x(t)\|_{M^{\alpha,p}} \leq \left(\|Ax(t)\|_{M^{\alpha,p}} + \|L(x(t))\|_{M^{\alpha,p}} + \left\|L\left(C_3(t)\sup_{\vartheta\in[-\tau,0]}x(t+\vartheta)\right)\right\|_{M^{\alpha,p}}\right)\frac{1}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{d\tau}{\tau} + \|Bu(t)\|_{M^{\alpha,p}}\frac{1}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{d\tau}{\tau} + \|x(a)\|_{M^{\alpha,p}}$$

$$\|x(t)\|_{M^{\alpha,p}} \leq \left(\|A\|_{M^{\alpha,p}} + \|L(1 + C_3(t))\|_{M^{\alpha,p}}\right)\frac{1}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\sup_{\vartheta\in[-\tau,0]}x(t+\vartheta)\frac{d\tau}{\tau} + \|Bu(t)\|_{M^{\alpha,p}}\frac{1}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{d\tau}{\tau} + x(a)$$

Set $\delta(t) = \|Bu(t)\|_{M^{\alpha,p}}\frac{1}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{d\tau}{\tau}$, we have that

$\delta(t) = \left(\|Bu(t)\|_p + \left\|{}^{RH}_{a}D_t^{\alpha,\mu}Bu(t)\right\|_p\right)\frac{1}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{d\tau}{\tau}$, we obtain that

$\delta(t) = \left(\|Bu(t)\|_p + \left\| \frac{\left(\log\frac{t}{a}\right)^{-\alpha}}{\Gamma(1-\alpha)} Bu(t) \right\|_p \right) \frac{-\left(\log\frac{t}{a}\right)^{\alpha}}{\alpha\Gamma(\alpha)}$, implies that

$\delta(t) = \left(\frac{\left(\log\frac{t}{a}\right)^{\alpha}}{\alpha\Gamma(\alpha)} \|Bu(t)\|_p + \|Bu(t)\|_p \frac{1}{\Gamma(1-\alpha)\alpha\Gamma(\alpha)} \right)$

$\sigma(t) = \left(\|A\|_{M^{\alpha,p}} + \left\|L\left(1 + C_3(t)\right)\right\|_{M^{\alpha,p}} \right) + \|x(a)\|_{M^{\alpha,p}}$

$= \left\|A + \left(L + LC_3(t)\right)\right\|_{M^{\alpha,p}} + \|x(a)\|_{M^{\alpha,p}}$

$\sigma(t) \le \|A\|_{M^{\alpha,p}} + \|L\|_{M^{\alpha,p}} + \|LC_3(t)\|_{M^{\alpha,p}} + \|x(a)\|_{M^{\alpha,p}}$, which given that

$\sigma(t) \le \|A\|_p + \|LC_3(t)\|_p + \|L\|_p + \left\|{}^{RH}_{a}D_t^{\alpha,\mu} A\right\|_p + \left\|{}^{RH}_{a}D_t^{\alpha,\mu} L\right\|_p + \left\|{}^{RH}_{a}D_t^{\alpha,\mu} LC_3(t)\right\|_p$

$\sigma(t) \le \|A\|_p + L \left\| \frac{\mu^{\alpha}(t^{\mu}-a^{\mu})}{\Gamma(1-\alpha)} \right\|_p + \|L\|_p + \left\| \frac{\left(\log\frac{t}{a}\right)^{-\alpha}}{\Gamma(1-\alpha)} A \right\|_P + L \left\| \frac{\left(\log\frac{t}{a}\right)^{-\alpha}}{\Gamma(1-\alpha)} \right\|_P +$

$L \left\| \left(\frac{\mu^{\alpha}(t^{\mu}-a^{\mu})}{\Gamma(1-\alpha)} \right) \left(\frac{\left(\log\frac{t}{a}\right)^{-\alpha}}{\Gamma(1-\alpha)} \right) \right\|_P + \|x(a)\|_p + \left\|{}^{RH}_{a}D_t^{\alpha,\mu} x(a)\right\|_p$, we get that

$\sigma(t) \le \|A\|_p + \|L\|_p \left\| \frac{\mu^{\alpha}(t^{\mu}-a^{\mu})}{\Gamma(1-\alpha)} \right\|_p + \|L\|_p + \left\| \frac{\left(\log\frac{t}{a}\right)^{-\alpha}}{\Gamma(1-\alpha)} A \right\|_P + \|L\|_p \left\| \frac{\left(\log\frac{t}{a}\right)^{-\alpha}}{\Gamma(1-\alpha)} \right\|_P +$

$\|L\|_p \left\| \left(\frac{\mu^{\alpha}(t^{\mu}-a^{\mu})}{\Gamma(1-\alpha)} \right) \left(\frac{\left(\log\frac{t}{a}\right)^{-\alpha}}{\Gamma(1-\alpha)} \right) \right\|_P + (1 + \frac{\left(\log\frac{t}{a}\right)^{-\alpha}}{\Gamma(1-\alpha)}) \|x(a)\|_p$

$\sigma(t) \le \|A\|_p + \|L\|_p \left\| \frac{\mu^{\alpha}(t^{\mu}-a^{\mu})}{\Gamma(1-\alpha)} \right\|_p + \|L\|_p + \left\| \frac{\left(\log\frac{t}{a}\right)^{-\alpha}}{\Gamma(1-\alpha)} A \right\|_P + \|L\|_p \left\| \frac{\left(\log\frac{t}{a}\right)^{-\alpha}}{\Gamma(1-\alpha)} \right\|_P +$

$\|L\|_p \left\| \left(\frac{\mu^{\alpha}(t^{\mu}-a^{\mu})}{\Gamma(1-\alpha)} \right) \left(\frac{\left(\log\frac{t}{a}\right)^{-\alpha}}{\Gamma(1-\alpha)} \right) \right\|_P + (1 + \frac{\left(\log\frac{t}{a}\right)^{-\alpha}}{\Gamma(1-\alpha)}) \|x(a)\|_p$, implies that

$\|x(t)\| \le \delta(t) + \sigma(t) \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \sup_{\vartheta\in[-\tau,0]} x(t+\vartheta) \frac{d\tau}{\tau}$

$\|x(t)\| \le \delta(t) + \left(\sigma(t) \left({}^{RH}_{a}D_t^{-\alpha,\mu} \left\| \sup_{\vartheta\in[-\tau,0]} x(t+\vartheta) \right\|_{M^{\alpha,p}} \right) \right)$, hence

$\| x(t) \| \le \delta(t) E_{\alpha} \left(\sigma(t) \left(\log\frac{t}{a}\right)^{\alpha} \Gamma(\alpha) \right)$.

3.2 Stability of the Riemann-Hadamard and Riemann-Katugampola Fractional order nonlinear differential control nonlocal system by Using the step method.

Theorem (3.2.6): Assume that the Riemann-Hadamard and Riemann-Katugampola −Katugampola fractional order nonlinear differential control nonlocal system (1.1) satisfy the conditions in lemma (3.8). Then the solution of (1.1) is finite-time stable if the following conditions are satisfied:

$$\delta_T(\tau)\, E_{\alpha}\sigma_0(T) \left(\log\frac{T}{a}\right)^{\alpha} \le \varepsilon \text{, where}$$

$\delta_T(\tau) = \delta_1(T) + L \left[\frac{1}{\Gamma(1+\alpha)} (\log j\tau)^{\alpha} \left(\sum_{j=1}^{n} \delta_j(\tau) E_{\alpha}\sigma_0(j\tau) \left((\tau^{\mu})^{\alpha} - (a^{\mu})^{\alpha}\right) \right) + \right.$
$\left. \delta_{n+1}(\tau) E_{\alpha}\sigma_0(n+1) \left(\log\frac{(n+1)\tau}{a}\right)^{\alpha} \frac{\mu^{\alpha}}{\Gamma(1-\alpha)} \left((T^{\mu} - (n+1)(\tau)^{\mu})^{\alpha}\right) \right] \delta_{i+1}(\tau) = \delta_1\left((i+1)\tau\right) +$
$L \left[\frac{\mu^{\alpha}}{\Gamma(1-\alpha)} (\tau^{\mu} - a^{\mu})^{\alpha} \left[\sum_{j=1}^{i} \delta_j(\tau) E_{\alpha}\sigma_0(j\tau) \left(\log\frac{\tau}{a}\right)^{\alpha} \right] \right]$

$\delta_1(t) = (\|Bu(t)\|_{M^{\alpha,p}} + LC_3(t)\|\vartheta\|_{M^{\alpha,p}})\frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \frac{d\tau}{\tau} + x(a)$

Proof

From formula (3.9)

$x(t) = {}^{RH}_{a}D_t^{-\alpha,\mu}\left(Ax(t) + f\left(t, x(t), {}^{RK}_{a}D_t^{\alpha,\mu}x(t-s)\right) + Bu(t)\right) + x(a) \quad t \in [0,\tau]$

Therefore,

$\|x(t)\|_{M^{\alpha,p}} \le \|Ax(t) + L(x(t))\|_{M^{\alpha,p}} \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \frac{d\tau}{\tau} + (\|Bu(t)\|_{M^{\alpha,p}} +$
$LC_3(t)x(t-s))\frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \frac{d\tau}{\tau} + \|x(a)\|_{M^{\alpha,p}}$

$\|x(t)\|_{M^{\alpha,p}} \le \|A + L\|_{M^{\alpha,p}} \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} x(\tau)\frac{d\tau}{\tau} + (\|Bu(t)\|_{M^{\alpha,p}} +$
$LC_3(t)\|\vartheta\|_{M^{\alpha,p}})\frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \frac{d\tau}{\tau} + \|x(a)\|_{M^{\alpha,p}}$

Set $\delta_1(t) = (\|Bu(t)\|_{M^{\alpha,p}} + LC_3(t)\|\vartheta\|_{M^{\alpha,p}})\frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \frac{d\tau}{\tau} + x(a)$

$\sigma_0(t) = \|A + L\|_{M^{\alpha,p}}$

By using the Lemma (3.5) for $t \in [0,\tau]$

$\|x(t)\|_{M^{\alpha,p}} \le \delta_1(t) + \sigma_0(t){}^{RH}_{a}D_t^{-\alpha}x(t)$ for $t \in [0,\tau]$, we have that

$\|x(t)\|_{M^{\alpha,p}} \le \delta_1(\tau)E_\alpha\sigma_0(\tau)\left(\log\frac{\tau}{\mathrm{a}}\right)^{\alpha}\Gamma(\alpha)$

For $t \in (i\tau, (i+1)\tau], 1 \le i \le n$, we have that

$\|x(t)\|_{M^{\alpha,p}} \le (\|A\|_{M^{\alpha,p}}\|x(t)\|_{M^{\alpha,p}} + L\|x(t)\|_{M^{\alpha,p}} + \|B\|_{M^{\alpha,p}}\|u(\tau)\|_{M^{\alpha,p}} +$
$+\|x(a)\|_{M^{\alpha,p}})\frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \frac{d\tau}{\tau} + \frac{L}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \|x(s-\tau)\|_{M^{\alpha,p}} \frac{d\tau}{\tau}$

$\|x(t)\|_{M^{\alpha,p}} \le (\|A\|_{M^{\alpha,p}}\|x(t)\|_{M^{\alpha,p}} + L\|x(t)\|_{M^{\alpha,p}} + \|B\|_{M^{\alpha,p}}\|u(\tau)\|_{M^{\alpha,p}} +$
$+\|x(a)\|_{M^{\alpha,p}})\frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \frac{d\tau}{\tau} + \frac{L\mu^\alpha}{\Gamma(1-\alpha)}(t^{1-\mu}\frac{d}{dt})\int_a^t \frac{\tau^{\mu-1}}{(t^\mu-\tau^\mu)^\alpha}\|x(s-\tau)\|_{M^{\alpha,p}} \frac{d\tau}{\tau}$, we obtain that

$\|x(t)\|_{M^{\alpha,p}} \le (\|A\|_{M^{\alpha,p}}\|x(t)\|_{M^{\alpha,p}} + L\|x(t)\|_{M^{\alpha,p}} + \|B\|_{M^{\alpha,p}}\|u(\tau)\|_{M^{\alpha,p}} +$
$+\|x(a)\|_{M^{\alpha,p}})\frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \frac{d\tau}{\tau} + L\left(\frac{\mu^\alpha}{\Gamma(1-\alpha)}(t^{1-\mu}\frac{d}{dt})\int_a^\tau \frac{s^{\mu-1}}{(t^\mu-s^\mu)^\alpha}\|x(s-\tau)\|_{M^{\alpha,p}} \frac{ds}{s} +\right.$
$\frac{\mu^\alpha}{\Gamma(1-\alpha)}(t^{1-\mu}\frac{d}{dt})\int_\tau^{2\tau} \frac{s^{\mu-1}}{(t^\mu-s^\mu)^\alpha}\|x(s-\tau)\|_{M^{\alpha,p}} \frac{ds}{s} + \cdots + \frac{\mu^\alpha}{\Gamma(1-\alpha)}(t^{1-\mu}\frac{d}{dt})\int_{i\tau}^t \frac{s^{\mu-1}}{(t^\mu-s^\mu)^\alpha}\|x(s-$
$\left.\tau)\|_{M^{\alpha,p}}\frac{ds}{s}\right)$, we get that

$\|x(t)\|_{M^{\alpha,p}} \le (\|A\|_{M^{\alpha,p}}\|x(t)\|_{M^{\alpha,p}} + L\|x(t)\|_{M^{\alpha,p}} + \|B\|_{M^{\alpha,p}}\|u(\tau)\|_{M^{\alpha,p}} +$
$+\|x(a)\|_{M^{\alpha,p}})\frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \frac{d\tau}{\tau} + L\left(\|x(s-\tau)\|_{M^{\alpha,p}}\frac{-\mu^\alpha}{\Gamma(1-\alpha)}((t^\mu-\tau^\mu)^\alpha -\right.$
$(t^\mu-a^\mu)^\alpha) + \|x(s-\tau)\|_{M^{\alpha,p}}\frac{-\mu^\alpha}{\Gamma(1-\alpha)}((t^\mu-2\tau^\mu)^\alpha - (t^\mu-\tau^\mu)^\alpha) + \cdots + \|x(s-$
$\left.\tau)\|_{M^{\alpha,p}}\frac{-\mu^\alpha}{\Gamma(1-\alpha)}((t^\mu-t^\mu)^\alpha - (t^\mu-i\tau^\mu)^\alpha)\right)$, we have that

$\|x(t)\|_{M^{\alpha,p}} \le (\|A\|_{M^{\alpha,p}}\|x(t)\|_{M^{\alpha,p}} + L\|x(t)\|_{M^{\alpha,p}} + \|B\|_{M^{\alpha,p}}\|u(\tau)\|_{M^{\alpha,p}} +$
$+\|x(a)\|_{M^{\alpha,p}})\frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \frac{d\tau}{\tau} + L\left(\|x(s-\tau)\|_{M^{\alpha,p}}\frac{\mu^\alpha}{\Gamma(1-\alpha)}((\tau^\mu-a^\mu)^\alpha) +\right.$
$\left.\|x(s-\tau)\|_{M^{\alpha,p}}\frac{\mu^\alpha}{\Gamma(1-\alpha)}((\tau)^\mu)^\alpha + \cdots + \|x(s-\tau)\|_{M^{\alpha,p}}\frac{\mu^\alpha}{\Gamma(1-\alpha)}((t^\mu-i\tau^\mu)^\alpha)\right)$

$\|x(t)\|_{M^{\alpha,p}} \le (\|A\|_{M^{\alpha,p}}\|x(t)\|_{M^{\alpha,p}} + L\|x(t)\|_{M^{\alpha,p}} + \|B\|_{M^{\alpha,p}}\|u(\tau)\|_{M^{\alpha,p}} +$
$+\|x(a)\|_{M^{\alpha,p}})\frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \frac{d\tau}{\tau} + L\left(\|\vartheta\|_{M^{\alpha,p}}\frac{\mu^\alpha}{\Gamma(1-\alpha)}((\tau^\mu-a^\mu)^\alpha) +\right.$

$\left(\delta_1(\tau)E_\alpha\sigma_0(\tau)\left(\log\frac{\tau}{a}\right)^\alpha\Gamma(\alpha)\right)\frac{\mu^\alpha}{\Gamma(1-\alpha)}((\tau)^\mu)^\alpha+\cdots+$

$\left.\left(\delta_i(\tau)E_\alpha\sigma_0(i\tau)\left(\log\frac{i\tau}{a}\right)^\alpha\Gamma(\alpha)\right)\frac{\mu^\alpha}{\Gamma(1-\alpha)}((t^\mu-i\tau^\mu)^\alpha)\right)$

$\|x(t)\|_{M^{\alpha,p}}\le(\|A\|_{M^{\alpha,p}}\|x(t)\|_{M^{\alpha,p}}+L\|x(t)\|_{M^{\alpha,p}}+\|B\|_{M^{\alpha,p}}\|u(\tau)\|_{M^{\alpha,p}}+$

$\|x(a)\|_{M^{\alpha,p}})\frac{1}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{d\tau}{\tau}+L\left(\omega\frac{\mu^\alpha}{\Gamma(1-\alpha)}((\tau^\mu-a^\mu)^\alpha)+\right.$

$\left(\delta_1(\tau)E_\alpha\sigma_0(\tau)\left(\log\frac{\tau}{a}\right)^\alpha\Gamma(\alpha)\right)\frac{\mu^\alpha}{\Gamma(1-\alpha)}((\tau)^\mu)^\alpha+\cdots+$

$\left.\left(\delta_i(\tau)E_\alpha\sigma_0(i\tau)\left(\log\frac{i\tau}{a}\right)^\alpha\Gamma(\alpha)\right)\frac{\mu^\alpha}{\Gamma(1-\alpha)}((t^\mu-i\tau^\mu)^\alpha)\right)$, implies that

$\|x(t)\|_{M^{\alpha,p}}\le(\|A\|_{M^{\alpha,p}}\|x(t)\|_{M^{\alpha,p}}+L\|x(t)\|_{M^{\alpha,p}}+\|B\|_{M^{\alpha,p}}\|u(\tau)\|_{M^{\alpha,p}}+$

$\|x(a)\|_{M^{\alpha,p}})\frac{1}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{d\tau}{\tau}+L\left(\omega\frac{\mu^\alpha}{\Gamma(1-\alpha)}((\tau^\mu-a^\mu)^\alpha)+\right.$

$\left(\delta_1(\tau)E_\alpha\sigma_0(\tau)\left(\log\frac{\tau}{a}\right)^\alpha\Gamma(\alpha)\right)\frac{\mu^\alpha}{\Gamma(1-\alpha)}((\tau)^\mu)^\alpha+\cdots+$

$\left.\left(\delta_i(\tau)E_\alpha\sigma_0(i\tau)\left(\log\frac{i\tau}{a}\right)^\alpha\Gamma(\alpha)\right)\frac{\mu^\alpha}{\Gamma(1-\alpha)}((t^\mu-i\tau^\mu)^\alpha)\right)$, we get that

$\delta_1(t)=(\|B\|_{M^{\alpha,p}}\|u(\tau)\|_{M^{\alpha,p}}+\|x(a)\|_{M^{\alpha,p}})\frac{1}{\Gamma(1+\alpha)}\left(\log\frac{\tau}{a}\right)^\alpha+L\left(\omega\frac{\mu^\alpha}{\Gamma(1-\alpha)}((\tau^\mu-a^\mu)^\alpha)\right).$

$\sigma_0(t)=(\|A\|_{M^{\alpha,p}}+L)$, hence

$\|x(t)\|_{M^{\alpha,p}}\le(\delta_1(t))+L\left(\left(\delta_1(\tau)E_\alpha\sigma_0(\tau)\left(\log\frac{\tau}{a}\right)^\alpha\Gamma(\alpha)\right)\frac{\mu^\alpha}{\Gamma(1-\alpha)}(\tau^\mu-a^\mu)^\alpha)+\cdots+\right.$

$\left.\left(\delta_i(\tau)E_\alpha\sigma_0(i\tau)(\log\frac{i\tau}{a})^\alpha\Gamma(\alpha)\right)\frac{\mu^\alpha}{\Gamma(1-\alpha)}((t^\mu-i\tau^\mu)^\alpha)\right)+\sigma_0(t)\,{}^{RK}_{a}D_t^{-\alpha,\mu}\|x(t)\|_{M^{\alpha,p}}$

$\|x(t)\|_{M^{\alpha,p}}\le\delta_1((i+1)\tau)+L\left[\frac{\mu^\alpha}{\Gamma(1-\alpha)}(\tau^\mu-a^\mu)^\alpha\left[\sum_{j=1}^i\delta_j(\tau)E_\alpha\sigma_0(j\tau)\left(\log\frac{\tau}{a}\right)^\alpha\Gamma(\alpha)\right]\right]+$

$\sigma_0((i+1)\tau)\,{}^{RK}_{a}D_t^{-\alpha,\mu}\|x(t)\|_{M^{\alpha,p}}$

Set $\delta_{i+1}(\tau)=\delta_1((i+1)\tau)+L\left[\frac{\mu^\alpha}{\Gamma(1-\alpha)}(\tau^\mu-a^\mu)^\alpha\left[\sum_{j=1}^i\delta_j(\tau)E_\alpha\sigma_0(j\tau)\left(\log\frac{\tau}{a}\right)^\alpha\Gamma(\alpha)\right]\right]$

By using lemma (3.5) , we obtain, for $t\in(i\tau,(i+1)\tau]$, implies that

$\|x(t)\|_{M^{\alpha,p}}\le\delta_{i+1}(\tau)\,E_\alpha\sigma_0((i+1)\tau)\left(\log\frac{t}{a}\right)^\alpha\Gamma(\alpha)$

$\|x(t)\|\le\delta_{i+1}(\tau)\,E_\alpha\sigma_0((i+1)\tau)\left(\log\frac{(i+1)\tau}{a}\right)^\alpha\Gamma(\alpha)$

Finally for $t\in((n+1)\tau,T],1\le i\le n$, we get that

$\|x(t)\|\le(\|A\|_{M^{\alpha,p}}\|x(t)\|_{M^{\alpha,p}}+L\,\|x(t)\|_{M^{\alpha,p}}+$

$\|B\|_{M^{\alpha,p}}\|u(\tau)\|_{M^{\alpha,p}})\frac{1}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{d\tau}{\tau}+\frac{L}{\Gamma(\alpha)}\int_a^t\left(\frac{t^\mu-\tau^\mu}{\mu}\right)^{\alpha-1}\|x(t-s)\|_{M^{\alpha,p}}\frac{d\tau}{\tau^{1-\mu}}$

$\|x(t)\|_{M^{\alpha,p}}\le(\|A\|_{M^{\alpha,p}}\|x(t)\|_{M^{\alpha,p}}+L\|x(t)\|_{M^{\alpha,p}}+\|B\|_{M^{\alpha,p}}\|u(\tau)\|_{M^{\alpha,p}}+$

$+\|x(a)\|_{M^{\alpha,p}})\frac{1}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{d\tau}{\tau}+L\left(\frac{\mu^\alpha}{\Gamma(1-\alpha)}\left(t^{1-\mu}\frac{d}{dt}\right)\int_a^\tau\frac{s^{\mu-1}}{(t^\mu-s^\mu)^\alpha}\|x(s-\tau)\|_{M^{\alpha,p}}\frac{ds}{s}+\right.$

$\frac{\mu^\alpha}{\Gamma(1-\alpha)}\left(t^{1-\mu}\frac{d}{dt}\right)\int_\tau^{2\tau}\frac{s^{\mu-1}}{(t^\mu-s^\mu)^\alpha}\|x(s-\tau)\|_{M^{\alpha,p}}\frac{ds}{s}+\cdots+$

$\frac{\mu^{\alpha}}{\Gamma(1-\alpha)}\left(t^{1-\mu}\frac{d}{dt}\right)\int_{n\tau}^{(n+1)\tau}\frac{s^{\mu-1}}{(t^{\mu}-s^{\mu})^{\alpha}}\|x(s-\tau)\|_{M^{\alpha,p}}\frac{ds}{s}+\frac{\mu^{\alpha}}{\Gamma(1-\alpha)}(t^{1-\mu}\frac{d}{dt})\int_{(n+1)\tau}^{t}\frac{s^{\mu-1}}{(t^{\mu}-s^{\mu})^{\alpha}}\|x(s-\tau)\|_{M^{\alpha,p}}\frac{ds}{s}\Big)$, we get that

$$\|x(t)\|_{M^{\alpha,p}} \le (\|A\|_{M^{\alpha,p}}\|x(t)\|_{M^{\alpha,p}} + L\|x(t)\|_{M^{\alpha,p}} + \|B\|_{M^{\alpha,p}}\|u(\tau)\|_{M^{\alpha,p}} + +\|x(a)\|_{M^{\alpha,p}})\frac{1}{\Gamma(1+\alpha)}\left(\log\frac{t}{a}\right)^{\alpha}\frac{d\tau}{\tau}+L\Big(\frac{-\mu^{\alpha}}{\Gamma(1-\alpha)}((t^{\mu}-\tau^{\mu})^{\alpha}-(t^{\mu}-a^{\mu})^{\alpha})\|x(s-\tau)\|_{M^{\alpha,p}} + \|x(s-\tau)\|_{M^{\alpha,p}}\frac{-\mu^{\alpha}}{\Gamma(1-\alpha)}((t^{\mu}-2\tau^{\mu})^{\alpha}-(t^{\mu}-\tau^{\mu})^{\alpha})+\cdots+\|x(s-\tau)\|_{M^{\alpha,p}}\frac{-\mu^{\alpha}}{\Gamma(1-\alpha)}((t^{\mu}-(n+1)(\tau)^{\mu})^{\alpha}-(t^{\mu}-n(\tau)^{\mu})^{\alpha})+\|x(s-\tau)\|_{M^{\alpha,p}}\frac{-\mu^{\alpha}}{\Gamma(1-\alpha)}((t^{\mu}-(n+1)(\tau)^{\mu})^{\alpha}-(t^{\mu}-t^{\mu})^{\alpha})\Big)$$

, we get that

$$\|x(t)\|_{M^{\alpha,p}} \le (\|A\|_{M^{\alpha,p}}\|x(t)\|_{M^{\alpha,p}} + L\|x(t)\|_{M^{\alpha,p}} + \|B\|_{M^{\alpha,p}}\|u(\tau)\|_{M^{\alpha,p}} + +\|x(a)\|_{M^{\alpha,p}})\frac{1}{\Gamma(1+\alpha)}\left(\log\frac{t}{a}\right)^{\alpha}\frac{d\tau}{\tau}+L\Big(\|\vartheta\|_{M^{\alpha,p}}\frac{\mu^{\alpha}}{\Gamma(1-\alpha)}((\tau^{\mu})^{\alpha}-(a^{\mu})^{\alpha})\|x(s-\tau)\|_{M^{\alpha,p}} + \left(\delta_1(\tau)E_{\alpha}\sigma_0\left(\log\frac{\tau}{a}\right)^{\alpha}\Gamma(\alpha)\right)\frac{\mu^{\alpha}}{\Gamma(1-\alpha)}(\tau^{\mu})^{\alpha}+\cdots+[\delta_n(\tau)E_{\alpha}\sigma_0(\log\frac{n\tau}{a})^{\alpha}\Gamma(\alpha)]\frac{\mu^{\alpha}}{\Gamma(1-\alpha)}((n(\tau)^{\mu})^{\alpha})+[\delta_{n+1}(\tau)E_{\alpha}\sigma_0(n+1)\left(\log\frac{\tau}{a}\right)^{\alpha}\Gamma(\alpha))]\frac{-\mu^{\alpha}}{\Gamma(1-\alpha)}((n+1)((\tau)^{\mu})^{\alpha}-(t^{\mu})^{\alpha})\Big)$$

, we get that

$$\|x(t)\|_{M^{\alpha,p}} \le (\|A\|_{M^{\alpha,p}}\|x(t)\|_{M^{\alpha,p}} + L\|x(t)\|_{M^{\alpha,p}} + \|B\|_{M^{\alpha,p}}\|u(\tau)\|_{M^{\alpha,p}} + +\|x(a)\|_{M^{\alpha,p}})\frac{1}{\Gamma(1+\alpha)}\left(\log\frac{t}{a}\right)^{\alpha}\frac{d\tau}{\tau}+L\Big(\|\vartheta\|_{M^{\alpha,p}}\frac{\mu^{\alpha}}{\Gamma(1-\alpha)}((\tau^{\mu})^{\alpha}-(a^{\mu})^{\alpha}) + \left(\delta_1(\tau)E_{\alpha}\sigma_0\left(\log\frac{\tau}{a}\right)^{\alpha}\Gamma(\alpha)\right)\frac{\mu^{\alpha}}{\Gamma(1-\alpha)}(\tau^{\mu})^{\alpha}+\cdots+[\delta_n(\tau)E_{\alpha}\sigma_0(\log\frac{n\tau}{a})^{\alpha}\Gamma(\alpha)]\frac{\mu^{\alpha}}{\Gamma(1-\alpha)}((n(\tau)^{\mu})^{\alpha})+[\delta_{n+1}(\tau)E_{\alpha}\sigma_0(n+1)\left(\log\frac{\tau}{a}\right)^{\alpha}\Gamma(\alpha))]\frac{-\mu^{\alpha}}{\Gamma(1-\alpha)}((t^{\mu}-(n+1)(\tau)^{\mu})^{\alpha})\Big)$$

, we get that

$$\|x(t)\| \le (\|B\|_{M^{\alpha,p}}\|u(\tau)\|_{M^{\alpha,p}}))\frac{1}{\Gamma(1+\alpha)}\left(\log\frac{\tau}{a}\right)^{\alpha}+L\left(\|\vartheta\|_{M^{\alpha,p}}\frac{\mu^{\alpha}}{\Gamma(1-\alpha)}((\tau^{\mu})^{\alpha}-(a^{\mu})^{\alpha})\right)+ ((\|A\|_{M^{\alpha,p}}+L)^{RK}_{a}D_t^{-\alpha}\|x(t)\|_{M^{\alpha,p}})+L\Big(\left(\delta_1(\tau)E_{\alpha}\sigma_0\left(\log\frac{\tau}{a}\right)^{\alpha}\Gamma(\alpha)\right)\frac{\mu^{\alpha}}{\Gamma(1-\alpha)}(\tau^{\mu})^{\alpha}+\cdots+[\delta_n(\tau)E_{\alpha}\sigma_0(\log\frac{n\tau}{a})^{\alpha}\Gamma(\alpha)]\frac{\mu^{\alpha}}{\Gamma(1-\alpha)}((n(\tau)^{\mu})^{\alpha})+[\delta_{n+1}(\tau)E_{\alpha}\sigma_0(n+1)\left(\log\frac{\tau}{a}\right)^{\alpha})\Gamma(\alpha)]\frac{\mu^{\alpha}}{\Gamma(1-\alpha)}((t^{\mu}-(n+1)(\tau)^{\mu})^{\alpha})\Big).$$

$.\delta_1(t)=(\|B\|_{M^{\alpha,p}}\|u(\tau)\|_{M^{\alpha,p}})\frac{1}{\Gamma(1+\alpha)}\left(\log\frac{\tau}{a}\right)^{\alpha}+L\left(\|\vartheta\|_{M^{\alpha,p}}\frac{\mu^{\alpha}}{\Gamma(1-\alpha)}((\tau^{\mu})^{\alpha}-(a^{\mu})^{\alpha})\right)$, $\sigma_0(t)=((\|A\|_{M^{\alpha,p}}+L))$

$$\|x(t)\| \le \delta_1(T)+L\left[\frac{1}{\Gamma(1+\alpha)}(\log j\tau)^{\alpha}\left(\sum_{j=1}^{n}\delta_j(\tau)E_{\alpha}\sigma_0(j\tau)\left(\log\frac{j\tau}{a}\right)^{\alpha}\Gamma(\alpha)((\tau^{\mu})^{\alpha}-(a^{\mu})^{\alpha})\right)+\delta_{n+1}(\tau)E_{\alpha}\sigma_0(n+1)\left(\log\frac{(n+1)\tau}{a}\right)^{\alpha}\Gamma(\alpha)\frac{\mu^{\alpha}}{\Gamma(1-\alpha)}((T^{\mu}-(n+1)(\tau)^{\mu})^{\alpha})\right]+\delta_0(t)^{RK}_{a}D_t^{-\alpha,\mu}\|x(t)\|_{M^{\alpha,p}}$$

$Set\ \delta_T(\tau) = \delta_1(T) + L\left[\frac{1}{\Gamma(1+\alpha)}(\log j\tau)^\alpha\left(\sum_{j=1}^{n}\delta_j(\tau)E_\alpha\sigma_0(j\tau)\left(\log\frac{j\tau}{a}\right)^\alpha\Gamma(\alpha)((\tau^\mu)^\alpha-(a^\mu)^\alpha)\right)+\right.$
$\left.\delta_{n+1}(\tau)E_\alpha\sigma_0(n+1)\left(\log\frac{(n+1)\tau}{a}\right)^\alpha\Gamma(\alpha)\frac{\mu^\alpha}{\Gamma(1-\alpha)}((T^\mu-(n+1)(\tau)^\mu)^\alpha)\right].$

By using lemma (3.5), we have that, for $t \in ((n+1)\tau, T]$, implies that

$$\|x(t)\| \le \delta_T(\tau)\, E_\alpha\sigma_0(T)\left(\log\frac{T}{a}\right)^\alpha\Gamma(\alpha) \qquad (3.11)$$

Remark (3.1):
The Riemann-Hadamard and Riemann-Katugampola Fractional order nonlinear differential control nonlocal system (1.1) is finite-time stable if it satisfies the following condition:

$$\delta_T(\tau)\, E_\alpha\sigma_0(T)\left(\log\frac{T}{a}\right)^\alpha\Gamma(\alpha) \le \varepsilon \quad \text{,where} \quad \delta_T(\tau) = \left(\frac{\left(\log\frac{T}{a}\right)^\alpha}{\alpha\Gamma(\alpha)}\|Bu(T)\|_p + \|Bu(T)\|_p\frac{1}{\Gamma(1-\alpha)\alpha\Gamma(\alpha)}\right) \text{ If } T \in (0,\tau].$$

4. Illustrative example
In this section, we provide one example to illustrate the finite time stability of the Riemann-Hadamard and Riemann-Katugampola −Katugampola fractional order nonlinear differential control nonlocal system (1.1).

Example (4.1):
Consider the following Riemann–Hadamard and Riemann −Katugampola Fractional order nonlinear differential control nonlocal system (1.1) as follows :

$$ {}^{CK}_{\ a}D_t^{\alpha,\mu}x(t) = \begin{pmatrix}0 & 1\\ -2 & 0\end{pmatrix}\begin{pmatrix}\sin(t)\\ cos(t)\end{pmatrix} + \begin{pmatrix}t\sin(t)\ {}^{RK}_{\ a}D_t^{\alpha,\mu}(\sin(t-0.1))\\ tcos(t)\ {}^{RK}_{\ a}D_t^{\alpha,\mu}(cos(t-0.1))\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}u(t), t\in[0,1]$$
(4.12)

where $u(t) = [u_1(t), u_2(t)]^T$ is a vector control functions. We have that $\|A\| = 2$, and. By using inequality (3.6) in Theorem (3.1.5) with $\alpha \in (0,1)$, and $L_1 = L_2 = 1$, $a = 0.1$, for $t \in [-\tau, 0]$, $\|B\| = 1$, $\|u(t)\| = 1$, we have that

Now we need to compute $\delta(t)\, E_\alpha\sigma(t)\left(\log\frac{t}{a}\right)^\alpha$ as follows $\delta(t) = \left(\frac{\left(\log\frac{t}{a}\right)^\alpha}{\alpha\Gamma(\alpha)}\|Bu(t)\|_p + \|Bu(t)\|_p\frac{1}{\Gamma(1-\alpha)\alpha\Gamma(\alpha)}\right)$ and

$$\sigma(t) \le \|A\|_p + \|L\|_p\left\|\frac{\mu^\alpha(t^\mu-a^\mu)}{\Gamma(1-\alpha)}\right\|_p + \|L\|_p + \left\|\frac{\left(\log\frac{t}{a}\right)^{-\alpha}}{\Gamma(1-\alpha)}A\right\|_P + \|L\|_p\left\|\frac{\left(\log\frac{t}{a}\right)^{-\alpha}}{\Gamma(1-\alpha)}\right\|_P +$$
$$\|L\|_p\left\|\left(\frac{\mu^\alpha(t^\mu-a^\mu)}{\Gamma(1-\alpha)}\right)\left(\frac{\left(\log\frac{t}{a}\right)^{-\alpha}}{\Gamma(1-\alpha)}\right)\right\|_P + \left(1+\frac{\left(\log\frac{t}{a}\right)^{-\alpha}}{\Gamma(1-\alpha)}\right)\|x(a)\|_p,$$

$\|\ x(t)\ \| \le \delta(t)E_{\alpha,\mu}\left(\sigma(t)\left(\log\frac{t}{a}\right)^\alpha\right)$. Therefore

$$\|\ x(t)\ \| \le \left(\left\|\frac{\left(\log\frac{t}{a}\right)^\alpha}{\alpha\Gamma(\alpha)} + \frac{1}{\Gamma(1-\alpha)\alpha\Gamma(\alpha)}\right\|_p\right)\sum_{i=0}^{\infty}\left(2 + \left\|\frac{\mu i^{i\alpha}(t^{\mu i}-a^{\mu i})}{\Gamma(1-\alpha i)}\right\|_p + 1 + 2\left\|\frac{\left(\log\frac{t}{a}\right)^{-\alpha i}}{\Gamma(1-\alpha i)}\right\|_P + \right.$$
$$\left.\left\|\frac{\left(\log\frac{t}{a}\right)^{-\alpha i}}{\Gamma(1-\alpha i)}\right\|_P + \left\|\left(\frac{\mu i^{\alpha i}(t^{\mu i}-a^{\mu})}{\Gamma(1-\alpha i)}\right)\left(\frac{\left(\log\frac{t}{a}\right)^{-\alpha i}}{\Gamma(1-\alpha i)}\right)\right\|_P + \left(1+\frac{\left(\log\frac{t}{a}\right)^{-\alpha i}}{\Gamma(1-\alpha i)}\right)\left\|\begin{pmatrix}\sin(a)\\ cos(a)\end{pmatrix}\right\|_p\right)\frac{\mathbf{1}}{\Gamma(i\alpha+\mu)} = \boldsymbol{\varepsilon}$$

Now the following tables and figures explained the values of epsilon depended on different values of and to be the Riemann-Hadamard and Riemann-Katugampola Fractional order nonlinear differential control nonlocal system (4.12) is stable with their different values of epsilon.

Table1. The value of ε, for $\alpha = 0.1$,

t	$\mu = \alpha$	$\mu = 0.2$	$\mu = 0.3$	$\mu = 0.4$	$\mu = 0.5$	$\mu = 0.6$	$\mu = 0.7$	$\mu = 0.8$	$\mu = 0.9$
0.2	1.5222	3.1704	4.8799	6.5917	8.2535	9.8214	11.2600	12.5421	13.6484
0.4	1.5909	3.3344	5.1563	6.9883	8.7704	10.4520	11.9929	13.3624	14.5395
0.6	1.6209	3.4120	5.2956	7.1991	9.0582	10.8185	12.4357	13.8764	15.1168
0.8	1.6400	3.4639	5.3924	7.3512	9.2733	11.1016	12.7890	14.2993	15.6065
1	1.6539	3.5030	5.4680	7.4733	9.4508	11.3413	13.0959	14.6762	16.0540

Table2. The value of ε, for $\alpha = 0.5$

t	$\mu = 0.1$	$\mu = 0.2$	$\mu = 0.3$	$\mu = 0.4$	$\mu = \alpha$	$\mu = 0.6$	$\mu = 0.7$	$\mu = 0.8$	$\mu = 0.9$
0.2	1.2014	2.5022	3.8514	5.2024	6.5139	7.7514	8.8868	9.8987	10.7718
0.4	1.5106	3.1661	4.8960	6.6355	8.3276	9.9244	11.3874	12.6878	13.8055
0.6	1.6589	3.4920	5.4196	7.3677	9.2704	11.0719	12.7270	14.2014	15.4709
0.8	1.7557	3.7082	5.7728	7.8697	9.9275	11.8847	13.6911	15.3080	16.7074
1	1.8271	3.8699	6.0406	8.2559	10.4405	12.5290	14.4673	16.2131	17.7352

Table3. The value of ε, for $\alpha = 0.9$

T	$\mu = 0.1$	$\mu = 0.2$	$\mu = 0.3$	$\mu = 0.4$	$\mu = 0.5$	$\mu = 0.6$	$\mu = 0.7$	$\mu = 0.8$	$\mu = \alpha$
0.2	0.6532	1.3604	2.0940	2.8285	3.5416	4.2144	4.8318	5.3819	5.8566
0.4	1.1564	2.4237	3.7480	5.0796	6.3749	7.5972	8.7172	9.7127	10.5683
0.6	1.4423	3.0361	4.7121	6.4058	8.0601	9.6265	11.0655	12.3474	13.4512
0.8	1.6432	3.4707	5.4030	7.3657	9.2916	11.1234	12.8141	14.3274	15.6372
1	1.7983	3.8089	5.9454	8.1258	10.2760	12.3315	14.2394	15.9576	17.4557

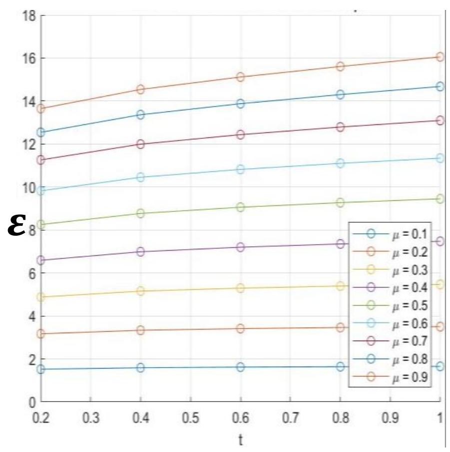

Figure 1.of table1.

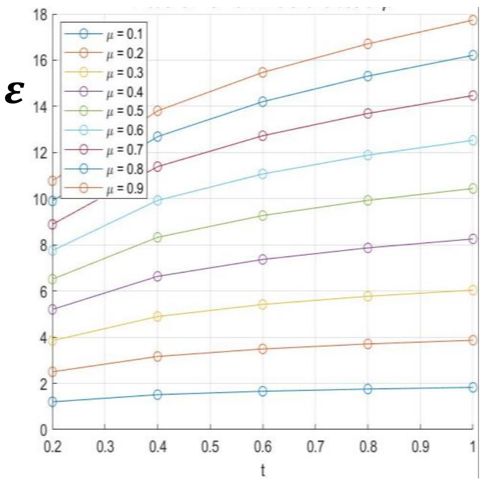

Figure 2. of table2.

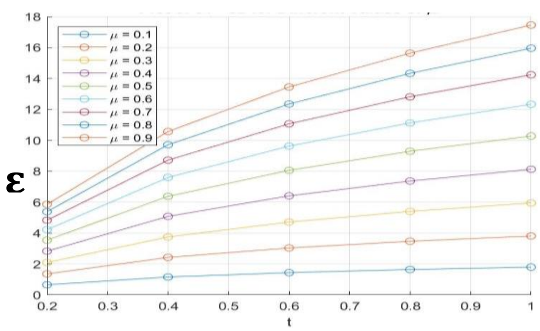

Figure 3. of table 3.

The graph displays the solution of the fractional differential equation over time for different values of the parameter T, highlighting the effect of these variations on the system's stability. As μ increases, the amplitude of $x(t)$ also rises for a given α, indicating that $x(t)$ grows with higher μ. When comparing the graphs for $\alpha = 0.1, \alpha = 0.5$, and $\alpha = 0.9$, it is clear that the overall amplitude of the function increases as μ grows, emphasizing the significant influence of α on the system's stability.

5. Conclusion

1. The Riemann–Hadamard and Riemann −Katugampola Fractional order nonlinear differential control nonlocal system (1.1) was very difficult for studyng since the fractional derivative types have expressions that are difficult to constrain.
2 .The uniqueness and existence on the generalized Gronwall inequality of the Riemann- Hadmard fractional derivative derivative, which was presented for the first time and played a good role in the stability of the system.
3 .The stability of finite time for our presented system was dependent on the maximal interval or on the step size of the maximal interval to obtain the guarantee estimation of epsilon.

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