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Using Triple Simple Elliptic Absolute Orlicz Function Defined by Triple Sequences Spaces $(\ell_\infty)^3_{\mathbb{F}}(\Theta)$ with Fuzzy Metric

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P A P E R I N F O

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A B S T R A C T

We present the triple simple elliptic absolute Orlicz function in this paper, which is determined by triple sequence spaces with fuzzy metrics. We also discuss some of its properties, such as that the space $(\ell_\infty)^3_{\mathbb{F}}(\Theta)$ is symmetric, solid and complete .

1. INTRODUCTION

L.A. Zadeh introduced fuzzy set theory in 1965, and a number of scholars have since adopted it, including Yu-ru [10], Tripathy and Baruah [1], Tripathy and Borgohain [3], Tripathy and Dutta [4], Tripathy and Sarma [7], [8], [9], and many more.

Kramosil and Michalek [6] created the fuzzy metric space by extending the idea of the probabilistic metric space to the fuzzy scene.

The space $(\ell_\infty)^3_{\mathbb{F}}(\Theta)$ produced by the basic elliptic absolute Orlicz function with fuzzy metric is defined and introduced in this study.

$\vartheta : [0,\infty) \to [0,\infty)$ is called an Orlicz function; it is a continuous, non-decreasing and convex function with $\vartheta(0) = 0, \vartheta(\mathfrak{A}) \succ 0$ as $\mathfrak{A} \succ 0$ and $\vartheta(\mathfrak{A}) \to \infty$.

2. DEFINITIONS AND PRELIMINARIES

A simple elliptic absolute Orlicz function is a function $\mathbb{M} : [0,\infty) \to [0,\infty) \ni \mathbb{M}(\mathfrak{A}) = -|\mathfrak{A}|^2\vartheta(\mathfrak{A})$, where ϑ is an Orlicz function.

A triple simple elliptic absolute Orlicz function is a function $\Theta : [0,\infty) \times [0,\infty) \times [0,\infty) \longrightarrow [0,\infty) \times [0,\infty) \times [0,\infty) \ni \Theta(\mathfrak{A},\mathfrak{S},\mathfrak{R}) = \big(\Theta_1(\mathfrak{A}),\Theta_2(\mathfrak{S}),\Theta_3(\mathfrak{R})\big)$, where $\Theta_1 : [0,\infty)\to[0,\infty) \ni \Theta_1(\mathfrak{A}) = -|\mathfrak{A}|^2\vartheta_1(\mathfrak{A})$, $\Theta_2 : [0,\infty)\to[0,\infty) \ni \Theta_2(\mathfrak{S}) = -|\mathfrak{S}|^2\vartheta_2(\mathfrak{S})$, $\Theta_3 : [0,\infty)\to[0,\infty) \ni \Theta_3(\mathfrak{R}) = -|\mathfrak{R}|^2\vartheta_3(\mathfrak{R})$. These functions are even, convex, continuous and non-decreasing , that hold the following conditions :

$\mathbb{i})\Theta_1(0) = 0, \Theta_2(0) = 0, \Theta_3(0) = 0 \Longrightarrow \Theta(0,0,0) = \big(\Theta_1(0),\Theta_2(0),\Theta_3(0)\big) = (0,0,0).$

$\mathbb{ii})\Theta_1(\mathfrak{A}) \succ 0 , \Theta_2(\mathfrak{S}) \succ 0 , \Theta_3(\mathfrak{R}) \succ 0 \Longrightarrow \Theta(\mathfrak{A} , \mathfrak{S} , \mathfrak{R}) = \big(\Theta_1(\mathfrak{A}),\Theta_2(\mathfrak{S}),\Theta_3(\mathfrak{R})\big) \succ (0,0,0)$, for $\mathfrak{A} \succ 0, \mathfrak{S} \succ 0, \mathfrak{R} \succ 0$,by which we say $(\mathfrak{A} , \mathfrak{S} , \mathfrak{R}) \succ (0,0,0)$ as $\Theta_1(\mathfrak{A}) \succ 0 , \Theta_2(\mathfrak{S}) \succ 0 , \Theta_3(\mathfrak{R}) \succ 0$.

$\mathbb{iii})\Theta_1(\mathfrak{A}) \to \infty, \Theta_2(\mathfrak{S}) \to \infty, \Theta_3(\mathfrak{R}) \to \infty$ as $\mathfrak{A} \to \infty, \mathfrak{S} \to \infty, \mathfrak{R} \to \infty \Longrightarrow \Theta(\mathfrak{A} , \mathfrak{S} , \mathfrak{R}) = \big(\Theta_1(\mathfrak{A}),\Theta_2(\mathfrak{S}),\Theta_3(\mathfrak{R})\big) \to (\infty,\infty,\infty)$ as $(\mathfrak{A} , \mathfrak{S} , \mathfrak{R}) \to (\infty,\infty,\infty)$ by which we say $\Theta(\mathfrak{A} , \mathfrak{S} , \mathfrak{R}) \to (\infty,\infty,\infty)$ as $\Theta_1(\mathfrak{A}) \to \infty, \Theta_2(\mathfrak{S}) \to \infty, \Theta_3(\mathfrak{R}) \to \infty$.

$\big(\ltimes_{\ell\hbar ji}\, \mathfrak{A}_{\ell\hbar j}\big) \in \mathbb{E}^3$ when $\big(\mathfrak{A}_{\ell\hbar j}\big) \in \mathbb{E}^3$ for every sequence of scalars with $\big|\ltimes_{\ell\hbar j}\big| \preccurlyeq 1, \forall\, \ell, \hbar, j \in \mathbb{N}$ implies that triple sequence space $\mathbb{E}^3$ is solid .

$\big(\mathfrak{A}_{\pi(\ell\hbar j)}\big) \in \mathbb{E}^3$ when $\big(\mathfrak{A}_{\ell\hbar ji}\big) \in \mathbb{E}^3$ leads to $\mathbb{E}^3$ is symmetric and π is a permutation of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$.

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$\mathcal{T} : \mathbb{R}(\mathbb{I}) \times \mathbb{R}(\mathbb{I}) \to \mathbb{R} \ni \mathcal{T}(\mathfrak{H},\mathfrak{G}) = \sup_{0<\ltimes\leq 1} \mathcal{T}_{\ltimes}(\mathfrak{H}^{\ltimes},\mathfrak{G}^{\ltimes}), \mathcal{T}_{\ltimes} : \mathbb{R}\times\mathbb{R} \to \mathbb{R} \ni \mathcal{T}_{\ltimes}(\mathfrak{H}^{\ltimes},\mathfrak{G}^{\ltimes}) = \min\{|\mathfrak{H}_1^{\ltimes} - \mathfrak{G}_1^{\ltimes}|, |\mathfrak{H}_2^{\ltimes} - \mathfrak{G}_2^{\ltimes}|\}$ and $\mathcal{S} : \mathbb{R}(\mathbb{I}) \times \mathbb{R}(\mathbb{I}) \to \mathbb{R} \ni \mathcal{S}(\mathfrak{H},\mathfrak{G}) = \sup_{0<\ltimes\leq 1} \mathcal{S}_{\ltimes}(\mathfrak{H}^{\ltimes},\mathfrak{G}^{\ltimes}), \mathcal{S}_{\ltimes} : \mathbb{R}\times\mathbb{R} \to \mathbb{R} \ni \mathcal{S}_{\ltimes}(\mathfrak{H}^{\ltimes},\mathfrak{G}^{\ltimes}) = \max\{|\mathfrak{H}_1^{\ltimes} - \mathfrak{G}_1^{\ltimes}|, |\mathfrak{H}_2^{\ltimes} - \mathfrak{G}_2^{\ltimes}|\}$

A Fuzzy metric space is the quadruple $(\mathbb{R}(\mathbb{I}), d_{\mathbb{F}}, \mathbb{T}, \mathbb{S})$, and $d_{\mathbb{F}}$ is a fuzzy metric if :

i) $d_{\mathbb{F}}(\mathfrak{E},\mathfrak{C}) = 0 \Leftrightarrow \mathfrak{E} = \mathfrak{C}, \forall\, \mathfrak{E},\mathfrak{C} \in \mathbb{R}(\mathbb{I})$.

ii) $d_{\mathbb{F}}(\mathfrak{E},\mathfrak{C}) = d_{\mathbb{F}}(\mathfrak{C},\mathfrak{E}), \forall\, \mathfrak{E},\mathfrak{C} \in \mathbb{R}(\mathbb{I})$.

iii) $\forall\, \mathfrak{E},\mathfrak{C},\mathfrak{G} \in \mathbb{R}(\mathbb{I})$,

a) $d_{\mathbb{F}}(\mathfrak{E},\mathfrak{C})(\mathfrak{Q}+\mathfrak{O}) \succcurlyeq \mathbb{T}\big(d_{\mathbb{F}}(\mathfrak{E},\mathfrak{G})(\mathfrak{Q}), d_{\mathbb{F}}(\mathfrak{G},\mathfrak{C})(\mathfrak{O})\big)$, whenever $\mathfrak{Q} \preccurlyeq \mathcal{T}_1(\mathfrak{E},\mathfrak{G}), \mathfrak{O} \preccurlyeq \mathcal{T}_1(\mathfrak{G},\mathfrak{C})$ and $\mathfrak{Q}+\mathfrak{O} \preccurlyeq \mathcal{T}_1(\mathfrak{E},\mathfrak{C})$.

b) $d_{\mathbb{F}}(\mathfrak{E},\mathfrak{C})(\mathfrak{Q}+\mathfrak{O}) \preccurlyeq \mathbb{T}\big(d_{\mathbb{F}}(\mathfrak{E},\mathfrak{G})(\mathfrak{Q}), d_{\mathbb{F}}(\mathfrak{G},\mathfrak{C})(\mathfrak{O})\big)$, whenever $\mathfrak{Q} \succcurlyeq \mathcal{T}_1(\mathfrak{E},\mathfrak{G}), \mathfrak{O} \succcurlyeq \mathcal{T}_1(\mathfrak{G},\mathfrak{C})$ and $\mathfrak{Q}+\mathfrak{O} \succcurlyeq \mathcal{T}_1(\mathfrak{E},\mathfrak{C})$.

$$(\ell_\infty)^3_{\mathbb{F}}(\Theta) = \Big\{\mathfrak{X}_{\mathfrak{abc}} = ((\mathfrak{X}_1)_{\mathfrak{abc}}, (\mathfrak{X}_2)_{\mathfrak{abc}}, (\mathfrak{X}_3)_{\mathfrak{abc}}) \in \mathbb{W}^3_{\mathbb{F}} : \sup_{\mathfrak{abc}}\Big[\Theta_1\Big(\frac{\mathcal{T}((\mathfrak{X}_1)_{\mathfrak{abc}},\bar{0})}{\rho}\Big) \curlyvee \Theta_2\Big(\frac{\mathcal{T}((\mathfrak{X}_2)_{\mathfrak{abc}},\bar{0})}{\rho}\Big) \curlyvee \Theta_3\Big(\frac{\mathcal{T}((\mathfrak{X}_3)_{\mathfrak{abc}},\bar{0})}{\rho}\Big)\Big] \prec (\infty,\infty,\infty) \text{ and } \sup_{\mathfrak{abc}}\Big[\Theta_1\Big(\frac{\mathcal{S}((\mathfrak{X}_1)_{\mathfrak{abc}},\bar{0})}{\rho}\Big) \curlyvee \Theta_2\Big(\frac{\mathcal{S}((\mathfrak{X}_2)_{\mathfrak{abc}},\bar{0})}{\rho}\Big) \curlyvee \Theta_3\Big(\frac{\mathcal{S}((\mathfrak{X}_3)_{\mathfrak{abc}},\bar{0})}{\rho}\Big)\Big] \prec (\infty,\infty,\infty), \text{for some } \rho \succ 0\Big\}$$
, where $\Theta = (\Theta_1,\Theta_2,\Theta_3)$.

3. MAIN RESULTS

Theorem 3.1:

$(\ell_\infty)^3_{\mathbb{F}}(\Theta)$ is metric space under the metric :

$$\bar{d}(\mathfrak{A},\mathfrak{S})_\Theta = \inf\Big\{(\rho,\rho,\rho) \succ (0,0,0) : \sup_{\mathfrak{abc}}\Big[\Theta_1\Big(\frac{\mathcal{T}((\mathfrak{A}_1)_{\mathfrak{abc}},(\mathfrak{S}_1)_{\mathfrak{abc}})}{\rho}\Big) \curlyvee \Theta_2\Big(\frac{\mathcal{T}((\mathfrak{A}_2)_{\mathfrak{abc}},(\mathfrak{S}_2)_{\mathfrak{abc}})}{\rho}\Big) \curlyvee \Theta_3\Big(\frac{\mathcal{T}((\mathfrak{A}_3)_{\mathfrak{abc}},(\mathfrak{S}_3)_{\mathfrak{abc}})}{\rho}\Big)\Big] \preccurlyeq (1,1,1) \text{ and } \sup_{\mathfrak{abc}}\Big[\Theta_1\Big(\frac{\mathcal{S}((\mathfrak{A}_1)_{\mathfrak{abc}},(\mathfrak{S}_1)_{\mathfrak{abc}})}{\rho}\Big) \curlyvee \Theta_2\Big(\frac{\mathcal{S}((\mathfrak{A}_2)_{\mathfrak{abc}},(\mathfrak{S}_2)_{\mathfrak{abc}})}{\rho}\Big) \curlyvee \Theta_3\Big(\frac{\mathcal{S}((\mathfrak{A}_3)_{\mathfrak{abc}},(\mathfrak{S}_3)_{\mathfrak{abc}})}{\rho}\Big)\Big] \preccurlyeq (1,1,1)\Big\}$$
$, \forall\, \mathfrak{A} = (\mathfrak{A}_1,\mathfrak{A}_2,\mathfrak{A}_3), \mathfrak{S} = (\mathfrak{S}_1,\mathfrak{S}_2,\mathfrak{S}_3) \in (\ell_\infty)^3_{\mathbb{F}}(\Theta)$.

Proof

$\forall\, \mathfrak{A},\mathfrak{S} \in (\ell_\infty)^3_{\mathbb{F}}(\Theta)$, we get,

i) If $\bar{d}(\mathfrak{A},\mathfrak{S})_\Theta = 0$.

This implies that

$\mathcal{T}((\mathfrak{A}_1)_{\mathfrak{abc}},(\mathfrak{S}_1)_{\mathfrak{abc}}) = 0, \mathcal{T}((\mathfrak{A}_2)_{\mathfrak{abc}},(\mathfrak{S}_2)_{\mathfrak{abc}}) = 0, \mathcal{T}((\mathfrak{A}_3)_{\mathfrak{abc}},(\mathfrak{S}_3)_{\mathfrak{abc}}) = 0$,

and

$\mathcal{S}((\mathfrak{A}_1)_{\mathfrak{abc}},(\mathfrak{S}_1)_{\mathfrak{abc}}) = 0, \mathcal{S}((\mathfrak{A}_2)_{\mathfrak{abc}},(\mathfrak{S}_2)_{\mathfrak{abc}}) = 0, \mathcal{S}((\mathfrak{A}_3)_{\mathfrak{abc}},(\mathfrak{S}_3)_{\mathfrak{abc}}) = 0$,

(since $\Theta_1(0) = 0, \Theta_2(0) = 0, \Theta_3(0) = 0$).

This indicate that , for all $\ltimes \in (0,1]$,

$\mathcal{T}((\mathfrak{A}_1)_{\mathfrak{abc}},(\mathfrak{S}_1)_{\mathfrak{abc}}) = \sup_{0<\ltimes\leq 1}\mathcal{T}_{\ltimes}((\mathfrak{A}_1)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{S}_1)^{\ltimes}_{\mathfrak{abc}}) = 0 \Rightarrow \mathcal{T}_{\ltimes}((\mathfrak{A}_1)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{S}_1)^{\ltimes}_{\mathfrak{abc}}) = 0, \forall \ltimes \in (0,1]$,

and

$\Rightarrow \min\{|(\mathfrak{A}_1)^{\ltimes}_{\mathfrak{abc}1} - (\mathfrak{S}_1)^{\ltimes}_{\mathfrak{abc}1}|, |(\mathfrak{A}_1)^{\ltimes}_{\mathfrak{abc}2} - (\mathfrak{S}_1)^{\ltimes}_{\mathfrak{abc}2}|\} = 0$ …. … … (1)

and

$\mathcal{T}((\mathfrak{A}_2)_{\mathfrak{abc}},(\mathfrak{S}_2)_{\mathfrak{abc}}) = \sup_{0<\ltimes\leq 1}\mathcal{T}_{\ltimes}((\mathfrak{A}_2)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{S}_2)^{\ltimes}_{\mathfrak{abc}}) = 0 \Rightarrow \mathcal{T}_{\ltimes}((\mathfrak{A}_2)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{S}_2)^{\ltimes}_{\mathfrak{abc}}) = 0$, for all $\ltimes \in (0,1]$

$\Rightarrow \min\{|(\mathfrak{A}_2)^{\ltimes}_{\mathfrak{abc}1} - (\mathfrak{S}_2)^{\ltimes}_{\mathfrak{abc}1}|, |(\mathfrak{A}_2)^{\ltimes}_{\mathfrak{abc}2} - (\mathfrak{S}_2)^{\ltimes}_{\mathfrak{abc}2}|\} = 0$, … … … (2)

and

$\mathcal{T}((\mathfrak{A}_3)_{\mathfrak{abc}},(\mathfrak{S}_3)_{\mathfrak{abc}}) = \sup_{0<\ltimes\leq 1}\mathcal{T}_{\ltimes}((\mathfrak{A}_3)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{S}_3)^{\ltimes}_{\mathfrak{abc}}) = 0 \Rightarrow \mathcal{T}_{\ltimes}((\mathfrak{A}_3)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{S}_3)^{\ltimes}_{\mathfrak{abc}}) = 0$, for all $\ltimes \in (0,1]$

$\Rightarrow \min\{|(\mathfrak{A}_3)^{\ltimes}_{\mathfrak{abc}1} - (\mathfrak{S}_3)^{\ltimes}_{\mathfrak{abc}1}|, |(\mathfrak{A}_3)^{\ltimes}_{\mathfrak{abc}2} - (\mathfrak{S}_3)^{\ltimes}_{\mathfrak{abc}2}|\} = 0, \forall \ltimes \in (0,1]$ … … … (3)

Similarly, for all $\ltimes \in (0,1]$,

$\mathcal{S}((\mathfrak{A}_1)_{\mathfrak{abc}},(\mathfrak{S}_1)_{\mathfrak{abc}}) = \sup_{0<\ltimes\leq 1}\mathcal{S}_{\ltimes}((\mathfrak{A}_1)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{S}_1)^{\ltimes}_{\mathfrak{abc}}) = 0 \Rightarrow \mathcal{S}_{\ltimes}((\mathfrak{A}_1)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{S}_1)^{\ltimes}_{\mathfrak{abc}}) = 0$, for all $\ltimes \in (0,1]$

and

$\Rightarrow \max\{|(\mathfrak{A}_1)^{\ltimes}_{\mathfrak{abc}1} - (\mathfrak{S}_1)^{\ltimes}_{\mathfrak{abc}1}|, |(\mathfrak{A}_1)^{\ltimes}_{\mathfrak{abc}2} - (\mathfrak{S}_1)^{\ltimes}_{\mathfrak{abc}2}|\} = 0$ … … (4)

and

$\mathcal{S}((\mathfrak{A}_2)_{\mathfrak{abc}},(\mathfrak{S}_2)_{\mathfrak{abc}}) = \sup_{0<\ltimes\leq 1}\mathcal{S}_{\ltimes}((\mathfrak{A}_2)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{S}_2)^{\ltimes}_{\mathfrak{abc}}) = 0 \Rightarrow \mathcal{S}_{\ltimes}((\mathfrak{A}_2)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{S}_2)^{\ltimes}_{\mathfrak{abc}}) = 0$, for all $\ltimes \in (0,1]$

$\Rightarrow \max\{|(\mathfrak{A}_2)^{\ltimes}_{\mathfrak{abc}1} - (\mathfrak{S}_2)^{\ltimes}_{\mathfrak{abc}1}|, |(\mathfrak{A}_2)^{\ltimes}_{\mathfrak{abc}2} - (\mathfrak{S}_2)^{\ltimes}_{\mathfrak{abc}2}|\} = 0$ … . … (5)

and

$\mathcal{S}((\mathfrak{A}_3)_{\mathfrak{abc}},(\mathfrak{S}_3)_{\mathfrak{abc}}) = \sup_{0<\ltimes\leq 1}\mathcal{S}_{\ltimes}((\mathfrak{A}_3)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{S}_3)^{\ltimes}_{\mathfrak{abc}}) = 0 \Rightarrow \mathcal{S}_{\ltimes}((\mathfrak{A}_3)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{S}_3)^{\ltimes}_{\mathfrak{abc}}) = 0, \forall \ltimes \in (0,1]$

$\Rightarrow \max\{|(\mathfrak{A}_3)^{\ltimes}_{\mathfrak{abc}1} - (\mathfrak{S}_3)^{\ltimes}_{\mathfrak{abc}1}|, |(\mathfrak{A}_3)^{\ltimes}_{\mathfrak{abc}2} - (\mathfrak{S}_3)^{\ltimes}_{\mathfrak{abc}2}|\} = 0$, for all $\ltimes \in (0,1]$ … … (6)

Based on (1), (2), (3), (4), (5), (6) , this indicates that , $(\mathfrak{A}_1)_{\mathfrak{abc}} = (\mathfrak{S}_1)_{\mathfrak{abc}}, (\mathfrak{A}_2)_{\mathfrak{abc}} = (\mathfrak{S}_2)_{\mathfrak{abc}}, (\mathfrak{A}_3)_{\mathfrak{abc}} = (\mathfrak{S}_3)_{\mathfrak{abc}} \Rightarrow \mathfrak{A} = \mathfrak{S}, \forall\, \mathfrak{a},\mathfrak{b},\mathfrak{c} \in \mathbb{N}$.

Conversely $\mathfrak{A} = \mathfrak{S}$.

Then, using $\mathcal{T}'$s , $\mathcal{S}'$s definitions , we obtain

$\mathcal{T}_{\ltimes}((\mathfrak{A}_1)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{S}_1)^{\ltimes}_{\mathfrak{abc}}) = 0, \mathcal{T}_{\ltimes}((\mathfrak{A}_2)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{S}_2)^{\ltimes}_{\mathfrak{abc}}) = 0, \mathcal{T}_{\ltimes}((\mathfrak{A}_3)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{S}_3)^{\ltimes}_{\mathfrak{abc}}) = 0$,

and

$\mathcal{S}_{\ltimes}((\mathfrak{A}_1)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{S}_1)^{\ltimes}_{\mathfrak{abc}}) = 0, \mathcal{S}_{\ltimes}((\mathfrak{A}_2)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{S}_2)^{\ltimes}_{\mathfrak{abc}}) = 0, \mathcal{S}_{\ltimes}((\mathfrak{A}_3)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{S}_3)^{\ltimes}_{\mathfrak{abc}}) = 0, \forall\, \mathfrak{a},\mathfrak{b},\mathfrak{c} \in \mathbb{N}$, for all $\ltimes \in (0,1]$.

This means that,

$\sup_{0<\ltimes\leq 1}\mathcal{T}_{\ltimes}((\mathfrak{A}_1)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{S}_1)^{\ltimes}_{\mathfrak{abc}}) = 0, \sup_{0<\ltimes\leq 1}\mathcal{T}_{\ltimes}((\mathfrak{A}_2)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{S}_2)^{\ltimes}_{\mathfrak{abc}}) = 0, \sup_{0<\ltimes\leq 1}\mathcal{T}_{\ltimes}((\mathfrak{A}_3)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{S}_3)^{\ltimes}_{\mathfrak{abc}}) = 0$

and

$\sup\limits_{0<\ltimes\leqslant 1}\mathcal{S}_{\ltimes}((\mathfrak{A}_1)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{S}_1)^{\ltimes}_{\mathfrak{abc}}) = 0,\ \sup\limits_{0<\ltimes\leqslant 1}\mathcal{S}_{\ltimes}((\mathfrak{A}_2)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{S}_2)^{\ltimes}_{\mathfrak{abc}}) = 0,\ \sup\limits_{0<\ltimes\leqslant 1}\mathcal{S}_{\ltimes}((\mathfrak{A}_3)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{S}_3)^{\ltimes}_{\mathfrak{abc}}) = 0, \forall\, \mathfrak{a},\mathfrak{b},\mathfrak{c}\in\mathbb{N}.$

Therefore $\mathcal{T}((\mathfrak{A}_1)_{\mathfrak{abc}},(\mathfrak{S}_1)_{\mathfrak{abc}}) = 0,\mathcal{T}((\mathfrak{A}_2)_{\mathfrak{abc}},(\mathfrak{S}_2)_{\mathfrak{abc}}) = 0,\mathcal{T}((\mathfrak{A}_3)_{\mathfrak{abc}},(\mathfrak{S}_3)_{\mathfrak{abc}}) = 0$ and

$\mathcal{S}((\mathfrak{A}_1)_{\mathfrak{abc}},(\mathfrak{S}_1)_{\mathfrak{abc}}) = 0,\mathcal{S}((\mathfrak{A}_2)_{\mathfrak{abc}},(\mathfrak{S}_2)_{\mathfrak{abc}}) = 0,\mathcal{S}((\mathfrak{A}_3)_{\mathfrak{abc}},(\mathfrak{S}_3)_{\mathfrak{abc}}) = 0$

Using Θ's continuity, we determine that $\bar{d}(\mathfrak{A},\mathfrak{S})_{\Theta} = 0$.

$\mathfrak{ii})$ $\bar{d}(\mathfrak{A},\mathfrak{S})_{\Theta} = \inf\Big\{(\rho,\rho,\rho)\succ(0,0,0):$ $\sup_{\mathfrak{abc}}\Big[\Theta_1\Big(\frac{\mathcal{T}((\mathfrak{A}_1)_{\mathfrak{abc}},(\mathfrak{S}_1)_{\mathfrak{abc}})}{\rho}\Big)\curlyvee\Theta_2\Big(\frac{\mathcal{T}((\mathfrak{A}_2)_{\mathfrak{abc}},(\mathfrak{S}_2)_{\mathfrak{abc}})}{\rho}\Big)\curlyvee\Theta_3\Big(\frac{\mathcal{T}((\mathfrak{A}_3)_{\mathfrak{abc}},(\mathfrak{S}_3)_{\mathfrak{abc}})}{\rho}\Big)\Big]\preccurlyeq(1,1,1)$ and $\sup_{\mathfrak{abc}}\Big[\Theta_1\Big(\frac{\mathcal{S}((\mathfrak{A}_1)_{\mathfrak{abc}},(\mathfrak{S}_1)_{\mathfrak{abc}})}{\rho}\Big)\curlyvee\Theta_2\Big(\frac{\mathcal{S}((\mathfrak{A}_2)_{\mathfrak{abc}},(\mathfrak{S}_2)_{\mathfrak{abc}})}{\rho}\Big)\curlyvee\Theta_3\Big(\frac{\mathcal{S}((\mathfrak{A}_3)_{\mathfrak{abc}},(\mathfrak{S}_3)_{\mathfrak{abc}})}{\rho}\Big)\Big]\preccurlyeq(1,1,1)\Big\}$.

Using $\mathcal{T}$'s definition, we have,

$\mathcal{T}((\mathfrak{A}_1)_{\mathfrak{abc}},(\mathfrak{S}_1)_{\mathfrak{abc}}) = \sup\limits_{0<\ltimes\leqslant 1}\mathcal{T}_{\ltimes}((\mathfrak{A}_1)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{S}_1)^{\ltimes}_{\mathfrak{abc}}) = \sup\limits_{0<\ltimes\leqslant 1}(\min\{|(\mathfrak{A}_1)^{\ltimes}_{\mathfrak{abc}1},(\mathfrak{S}_1)^{\ltimes}_{\mathfrak{abc}1}|,|(\mathfrak{A}_1)^{\ltimes}_{\mathfrak{abc}2},(\mathfrak{S}_1)^{\ltimes}_{\mathfrak{abc}2}|\}) = \sup\limits_{0<\ltimes\leqslant 1}(\min\{|(\mathfrak{S}_1)^{\ltimes}_{\mathfrak{abc}1},(\mathfrak{A}_1)^{\ltimes}_{\mathfrak{abc}1}|,|(\mathfrak{S}_1)^{\ltimes}_{\mathfrak{abc}2},(\mathfrak{A}_1)^{\ltimes}_{\mathfrak{abc}2}|\}) = \sup\limits_{0<\ltimes\leqslant 1}\mathcal{T}_{\ltimes}((\mathfrak{S}_1)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{A}_1)^{\ltimes}_{\mathfrak{abc}}) = \mathcal{T}((\mathfrak{S}_1)_{\mathfrak{abc}},(\mathfrak{A}_1)_{\mathfrak{abc}}),$

and

$\mathcal{T}((\mathfrak{A}_2)_{\mathfrak{abc}},(\mathfrak{S}_2)_{\mathfrak{abc}}) = \sup\limits_{0<\ltimes\leqslant 1}\mathcal{T}_{\ltimes}((\mathfrak{A}_2)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{S}_2)^{\ltimes}_{\mathfrak{abc}}) = \sup\limits_{0<\ltimes\leqslant 1}(\min\{|(\mathfrak{A}_2)^{\ltimes}_{\mathfrak{abc}1},(\mathfrak{S}_2)^{\ltimes}_{\mathfrak{abc}1}|,|(\mathfrak{A}_2)^{\ltimes}_{\mathfrak{abc}2},(\mathfrak{S}_2)^{\ltimes}_{\mathfrak{abc}2}|\})\ \sup\limits_{0<\ltimes\leqslant 1}(\min\{|(\mathfrak{S}_2)^{\ltimes}_{\mathfrak{abc}1},(\mathfrak{A}_2)^{\ltimes}_{\mathfrak{abc}1}|,|(\mathfrak{S}_2)^{\ltimes}_{\mathfrak{abc}2},(\mathfrak{A}_2)^{\ltimes}_{\mathfrak{abc}2}|\}) = \sup\limits_{0<\ltimes\leqslant 1}\mathcal{T}_{\ltimes}((\mathfrak{S}_2)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{A}_2)^{\ltimes}_{\mathfrak{abc}}) = \mathcal{T}((\mathfrak{S}_2)_{\mathfrak{abc}},(\mathfrak{A}_2)_{\mathfrak{abc}})$

and

$\mathcal{T}((\mathfrak{A}_3)_{\mathfrak{abc}},(\mathfrak{S}_3)_{\mathfrak{abc}}) = \sup\limits_{0<\ltimes\leqslant 1}\mathcal{T}_{\ltimes}((\mathfrak{A}_3)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{S}_3)^{\ltimes}_{\mathfrak{abc}}) = \sup\limits_{0<\ltimes\leqslant 1}(\min\{|(\mathfrak{A}_3)^{\ltimes}_{\mathfrak{abc}1},(\mathfrak{S}_2)^{\ltimes}_{\mathfrak{abc}1}|,|(\mathfrak{A}_3)^{\ltimes}_{\mathfrak{abc}2},(\mathfrak{S}_3)^{\ltimes}_{\mathfrak{abc}2}|\}) = \sup\limits_{0<\ltimes\leqslant 1}(\min\{|(\mathfrak{S}_3)^{\ltimes}_{\mathfrak{abc}1},(\mathfrak{A}_3)^{\ltimes}_{\mathfrak{abc}1}|,|(\mathfrak{S}_3)^{\ltimes}_{\mathfrak{abc}2},(\mathfrak{A}_3)^{\ltimes}_{\mathfrak{abc}2}|\}) = \sup\limits_{0<\ltimes\leqslant 1}\mathcal{T}_{\ltimes}((\mathfrak{S}_3)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{A}_3)^{\ltimes}_{\mathfrak{abc}}) = \mathcal{T}((\mathfrak{S}_3)_{\mathfrak{abc}},(\mathfrak{A}_3)_{\mathfrak{abc}}).$

Therefore $\mathcal{T}((\mathfrak{A}_1)_{\mathfrak{abc}},(\mathfrak{S}_1)_{\mathfrak{abc}}) = \mathcal{T}((\mathfrak{S}_1)_{\mathfrak{abc}},(\mathfrak{A}_1)_{\mathfrak{abc}}),\mathcal{T}((\mathfrak{A}_2)_{\mathfrak{abc}},(\mathfrak{S}_2)_{\mathfrak{abc}}) = \mathcal{T}((\mathfrak{S}_2)_{\mathfrak{abc}},(\mathfrak{A}_2)_{\mathfrak{abc}}),\mathcal{T}((\mathfrak{A}_3)_{\mathfrak{abc}},(\mathfrak{S}_3)_{\mathfrak{abc}}) = \mathcal{T}((\mathfrak{S}_1)_{\mathfrak{abc}},(\mathfrak{A}_1)_{\mathfrak{abc}})$.

By proceeding in the same manner, we obtain,

$\mathcal{S}((\mathfrak{A}_1)_{\mathfrak{abc}},(\mathfrak{S}_1)_{\mathfrak{abc}}) = \sup\limits_{0<\ltimes\leqslant 1}\mathcal{S}_{\ltimes}((\mathfrak{A}_1)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{S}_1)^{\ltimes}_{\mathfrak{abc}}) = \sup\limits_{0<\ltimes\leqslant 1}(\min\{|(\mathfrak{A}_1)^{\ltimes}_{\mathfrak{abc}1},(\mathfrak{S}_1)^{\ltimes}_{\mathfrak{abc}1}|,|(\mathfrak{A}_1)^{\ltimes}_{\mathfrak{abc}2},(\mathfrak{S}_1)^{\ltimes}_{\mathfrak{abc}2}|\}) = \sup\limits_{0<\ltimes\leqslant 1}(\min\{|(\mathfrak{S}_1)^{\ltimes}_{\mathfrak{abc}1},(\mathfrak{A}_1)^{\ltimes}_{\mathfrak{abc}1}|,|(\mathfrak{S}_1)^{\ltimes}_{\mathfrak{abc}2},(\mathfrak{A}_1)^{\ltimes}_{\mathfrak{abc}2}|\}) = \sup\limits_{0<\ltimes\leqslant 1}\mathcal{S}_{\ltimes}((\mathfrak{S}_1)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{A}_1)^{\ltimes}_{\mathfrak{abc}}) = \mathcal{S}((\mathfrak{S}_1)_{\mathfrak{abc}},(\mathfrak{A}_1)_{\mathfrak{abc}})$

and

$\mathcal{S}((\mathfrak{A}_2)_{\mathfrak{abc}},(\mathfrak{S}_2)_{\mathfrak{abc}}) = \sup\limits_{0<\ltimes\leqslant 1}\mathcal{S}_{\ltimes}((\mathfrak{A}_2)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{S}_2)^{\ltimes}_{\mathfrak{abc}}) = \sup\limits_{0<\ltimes\leqslant 1}(\min\{|(\mathfrak{A}_2)^{\ltimes}_{\mathfrak{abc}1},(\mathfrak{S}_2)^{\ltimes}_{\mathfrak{abc}1}|,|(\mathfrak{A}_2)^{\ltimes}_{\mathfrak{abc}2},(\mathfrak{S}_2)^{\ltimes}_{\mathfrak{abc}2}|\}) = \sup\limits_{0<\ltimes\leqslant 1}(\min\{|(\mathfrak{S}_2)^{\ltimes}_{\mathfrak{abc}1},(\mathfrak{A}_2)^{\ltimes}_{\mathfrak{abc}1}|,|(\mathfrak{S}_2)^{\ltimes}_{\mathfrak{abc}2},(\mathfrak{A}_2)^{\ltimes}_{\mathfrak{abc}2}|\}) = \sup\limits_{0<\ltimes\leqslant 1}\mathcal{S}_{\ltimes}((\mathfrak{S}_2)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{A}_2)^{\ltimes}_{\mathfrak{abc}}) = \mathcal{S}((\mathfrak{S}_2)_{\mathfrak{abc}},(\mathfrak{A}_2)_{\mathfrak{abc}})$

and

$\mathcal{S}((\mathfrak{A}_3)_{\mathfrak{abc}},(\mathfrak{S}_3)_{\mathfrak{abc}}) = \sup\limits_{0<\ltimes\leqslant 1}\mathcal{S}_{\ltimes}((\mathfrak{A}_3)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{S}_3)^{\ltimes}_{\mathfrak{abc}}) = \sup\limits_{0<\ltimes\leqslant 1}(\min\{|(\mathfrak{A}_3)^{\ltimes}_{\mathfrak{abc}1},(\mathfrak{S}_2)^{\ltimes}_{\mathfrak{abc}1}|,|(\mathfrak{A}_3)^{\ltimes}_{\mathfrak{abc}2},(\mathfrak{S}_3)^{\ltimes}_{\mathfrak{abc}2}|\}) = \sup\limits_{0<\ltimes\leqslant 1}(\min\{|(\mathfrak{S}_3)^{\ltimes}_{\mathfrak{abc}1},(\mathfrak{A}_3)^{\ltimes}_{\mathfrak{abc}1}|,|(\mathfrak{S}_3)^{\ltimes}_{\mathfrak{abc}2},(\mathfrak{A}_3)^{\ltimes}_{\mathfrak{abc}2}|\}) = \sup\limits_{0<\ltimes\leqslant 1}\mathcal{S}_{\ltimes}((\mathfrak{S}_3)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{A}_3)^{\ltimes}_{\mathfrak{abc}}) = \mathcal{S}((\mathfrak{S}_3)_{\mathfrak{abc}},(\mathfrak{A}_3)_{\mathfrak{abc}})$

Therefore $\mathcal{S}((\mathfrak{A}_1)_{\mathfrak{abc}},(\mathfrak{S}_1)_{\mathfrak{abc}}) = \mathcal{S}((\mathfrak{S}_1)_{\mathfrak{abc}},(\mathfrak{A}_1)_{\mathfrak{abc}}),\mathcal{S}((\mathfrak{A}_2)_{\mathfrak{abc}},(\mathfrak{S}_2)_{\mathfrak{abc}}) = \mathcal{S}((\mathfrak{S}_2)_{\mathfrak{abc}},(\mathfrak{A}_2)_{\mathfrak{abc}}),\mathcal{S}((\mathfrak{A}_3)_{\mathfrak{abc}},(\mathfrak{S}_3)_{\mathfrak{abc}}) = \mathcal{S}((\mathfrak{S}_1)_{\mathfrak{abc}},(\mathfrak{A}_1)_{\mathfrak{abc}})$.

Thus, we conclude

$$\bar{d}(\mathfrak{A},\mathfrak{S})_{\Theta} = \inf\Bigg\{(\rho,\rho,\rho)\succ(0,0,0) : \sup_{\mathfrak{abc}}\Bigg[\Theta_1\left(\frac{\mathcal{T}((\mathfrak{A}_1)_{\mathfrak{abc}},(\mathfrak{S}_1)_{\mathfrak{abc}})}{\rho}\right)\curlyvee\Theta_2\left(\frac{\mathcal{T}((\mathfrak{A}_2)_{\mathfrak{abc}},(\mathfrak{S}_2)_{\mathfrak{abc}})}{\rho}\right)\curlyvee\Theta_3\left(\frac{\mathcal{T}((\mathfrak{A}_3)_{\mathfrak{abc}},(\mathfrak{S}_3)_{\mathfrak{abc}})}{\rho}\right)\Bigg]\preccurlyeq(1,1,1) \text{ and } \sup_{\mathfrak{abc}}\Bigg[\Theta_1\left(\frac{\mathcal{S}((\mathfrak{A}_1)_{\mathfrak{abc}},(\mathfrak{S}_1)_{\mathfrak{abc}})}{\rho}\right)\curlyvee\Theta_2\left(\frac{\mathcal{S}((\mathfrak{A}_2)_{\mathfrak{abc}},(\mathfrak{S}_2)_{\mathfrak{abc}})}{\rho}\right)\curlyvee\Theta_3\left(\frac{\mathcal{S}((\mathfrak{A}_3)_{\mathfrak{abc}},(\mathfrak{S}_3)_{\mathfrak{abc}})}{\rho}\right)\Bigg]\preccurlyeq(1,1,1)\Bigg\}$$

$\sup\limits_{0<\ltimes\leqslant 1}(\min\{|(\mathfrak{S}_2)^{\ltimes}_{\mathfrak{abc}1},(\mathfrak{A}_2)^{\ltimes}_{\mathfrak{abc}1}|,|(\mathfrak{S}_2)^{\ltimes}_{\mathfrak{abc}2},(\mathfrak{A}_2)^{\ltimes}_{\mathfrak{abc}2}|\}) = \sup\limits_{0<\ltimes\leqslant 1}\mathcal{T}_{\ltimes}((\mathfrak{S}_2)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{A}_2)^{\ltimes}_{\mathfrak{abc}}) = \mathcal{T}((\mathfrak{S}_2)_{\mathfrak{abc}},(\mathfrak{A}_2)_{\mathfrak{abc}})$

and

$\mathcal{T}((\mathfrak{A}_3)_{\mathfrak{abc}},(\mathfrak{S}_3)_{\mathfrak{abc}}) = \sup\limits_{0<\ltimes\leqslant 1}\mathcal{T}_{\ltimes}((\mathfrak{A}_3)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{S}_3)^{\ltimes}_{\mathfrak{abc}}) = \sup\limits_{0<\ltimes\leqslant 1}(\min\{|(\mathfrak{A}_3)^{\ltimes}_{\mathfrak{abc}1},(\mathfrak{S}_2)^{\ltimes}_{\mathfrak{abc}1}|,|(\mathfrak{A}_3)^{\ltimes}_{\mathfrak{abc}2},(\mathfrak{S}_3)^{\ltimes}_{\mathfrak{abc}2}|\}) = \sup\limits_{0<\ltimes\leqslant 1}(\min\{|(\mathfrak{S}_3)^{\ltimes}_{\mathfrak{abc}1},(\mathfrak{A}_3)^{\ltimes}_{\mathfrak{abc}1}|,|(\mathfrak{S}_3)^{\ltimes}_{\mathfrak{abc}2},(\mathfrak{A}_3)^{\ltimes}_{\mathfrak{abc}2}|\}) = \sup\limits_{0<\ltimes\leqslant 1}\mathcal{T}_{\ltimes}((\mathfrak{S}_3)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{A}_3)^{\ltimes}_{\mathfrak{abc}}) = \mathcal{T}((\mathfrak{S}_3)_{\mathfrak{abc}},(\mathfrak{A}_3)_{\mathfrak{abc}}).$

Therefore $\mathcal{T}((\mathfrak{A}_1)_{\mathfrak{abc}},(\mathfrak{S}_1)_{\mathfrak{abc}}) = \mathcal{T}((\mathfrak{S}_1)_{\mathfrak{abc}},(\mathfrak{A}_1)_{\mathfrak{abc}}),\mathcal{T}((\mathfrak{A}_2)_{\mathfrak{abc}},(\mathfrak{S}_2)_{\mathfrak{abc}}) = \mathcal{T}((\mathfrak{S}_2)_{\mathfrak{abc}},(\mathfrak{A}_2)_{\mathfrak{abc}}),\mathcal{T}((\mathfrak{A}_3)_{\mathfrak{abc}},(\mathfrak{S}_3)_{\mathfrak{abc}}) = \mathcal{T}((\mathfrak{S}_1)_{\mathfrak{abc}},(\mathfrak{A}_1)_{\mathfrak{abc}})$.

By proceeding in the same manner, we obtain,

$\mathcal{S}((\mathfrak{A}_1)_{\mathfrak{abc}},(\mathfrak{S}_1)_{\mathfrak{abc}}) = \sup\limits_{0<\ltimes\leqslant 1}\mathcal{S}_{\ltimes}((\mathfrak{A}_1)^{\ltimes}_{\mathfrak{abc}},(\mathfrak{S}_1)^{\ltimes}_{\mathfrak{abc}}) = \sup\limits_{0<\ltimes\leqslant 1}(\min\{|(\mathfrak{A}_1)^{\ltimes}_{\mathfrak{abc}1},(\mathfrak{S}_1)^{\ltimes}_{\mathfrak{abc}1}|,|(\mathfrak{A}_1)^{\ltimes}_{\mathfrak{abc}2},(\mathfrak{S}_1)^{\ltimes}_{\mathfrak{abc}2}|\}) =$

$$\sup_{0<\ltimes\leqslant 1}(\min\{|(\mathfrak{S}_1)^{\ltimes}_{abc1},(\mathfrak{A}_1)^{\ltimes}_{abc1}|,|(\mathfrak{S}_1)^{\ltimes}_{abc2},(\mathfrak{A}_1)^{\ltimes}_{abc2}|\}) = \sup_{0<\ltimes\leqslant 1}\mathcal{S}_{\ltimes}((\mathfrak{S}_1)^{\ltimes}_{abc},(\mathfrak{A}_1)^{\ltimes}_{abc}) = \mathcal{S}((\mathfrak{S}_1)_{abc},(\mathfrak{A}_1)_{abc})$$

and

$$\mathcal{S}((\mathfrak{A}_2)_{abc},(\mathfrak{S}_2)_{abc}) = \sup_{0<\ltimes\leqslant 1}\mathcal{S}_{\ltimes}((\mathfrak{A}_2)^{\ltimes}_{abc},(\mathfrak{S}_2)^{\ltimes}_{abc}) = \sup_{0<\ltimes\leqslant 1}(\min\{|(\mathfrak{A}_2)^{\ltimes}_{abc1},(\mathfrak{S}_2)^{\ltimes}_{abc1}|,|(\mathfrak{A}_2)^{\ltimes}_{abc2},(\mathfrak{S}_2)^{\ltimes}_{abc2}|\}) = \sup_{0<\ltimes\leqslant 1}(\min\{|(\mathfrak{S}_2)^{\ltimes}_{abc1},(\mathfrak{A}_2)^{\ltimes}_{abc1}|,|(\mathfrak{S}_2)^{\ltimes}_{abc2},(\mathfrak{A}_2)^{\ltimes}_{abc2}|\}) = \sup_{0<\ltimes\leqslant 1}\mathcal{S}_{\ltimes}((\mathfrak{S}_2)^{\ltimes}_{abc},(\mathfrak{A}_2)^{\ltimes}_{abc}) = \mathcal{S}((\mathfrak{S}_2)_{abc},(\mathfrak{A}_2)_{abc})$$

and

$$\mathcal{S}((\mathfrak{A}_3)_{abc},(\mathfrak{S}_3)_{abc}) = \sup_{0<\ltimes\leqslant 1}\mathcal{S}_{\ltimes}((\mathfrak{A}_3)^{\ltimes}_{abc},(\mathfrak{S}_3)^{\ltimes}_{abc}) = \sup_{0<\ltimes\leqslant 1}(\min\{|(\mathfrak{A}_3)^{\ltimes}_{abc1},(\mathfrak{S}_2)^{\ltimes}_{abc1}|,|(\mathfrak{A}_3)^{\ltimes}_{abc2},(\mathfrak{S}_3)^{\ltimes}_{abc2}|\}) = \sup_{0<\ltimes\leqslant 1}(\min\{|(\mathfrak{S}_3)^{\ltimes}_{abc1},(\mathfrak{A}_3)^{\ltimes}_{abc1}|,|(\mathfrak{S}_3)^{\ltimes}_{abc2},(\mathfrak{A}_3)^{\ltimes}_{abc2}|\}) = \sup_{0<\ltimes\leqslant 1}\mathcal{S}_{\ltimes}((\mathfrak{S}_3)^{\ltimes}_{abc},(\mathfrak{A}_3)^{\ltimes}_{abc}) = \mathcal{S}((\mathfrak{S}_3)_{abc},(\mathfrak{A}_3)_{abc}).$$

Therefore $\mathcal{S}((\mathfrak{A}_1)_{abc},(\mathfrak{S}_1)_{abc}) = \mathcal{S}((\mathfrak{S}_1)_{abc},(\mathfrak{A}_1)_{abc}), \mathcal{S}((\mathfrak{A}_2)_{abc},(\mathfrak{S}_2)_{abc}) = \mathcal{S}((\mathfrak{S}_2)_{abc},(\mathfrak{A}_2)_{abc}), \mathcal{S}((\mathfrak{A}_3)_{abc},(\mathfrak{S}_3)_{abc}) = \mathcal{S}((\mathfrak{S}_1)_{abc},(\mathfrak{A}_1)_{abc})$.

Thus, we obtain that,

$$\bar{d}(\mathfrak{A},\mathfrak{S})_{\Theta} = \inf\Big\{(\rho,\rho,\rho) \succ (0,0,0): \sup_{abc}\Big[\Theta_1\Big(\frac{\mathcal{T}((\mathfrak{A}_1)_{abc},(\mathfrak{S}_1)_{abc})}{\rho}\Big) \curlyvee \Theta_2\Big(\frac{\mathcal{T}((\mathfrak{A}_2)_{abc},(\mathfrak{S}_2)_{abc})}{\rho}\Big) \curlyvee \Theta_3\Big(\frac{\mathcal{T}((\mathfrak{A}_3)_{abc},(\mathfrak{S}_3)_{abc})}{\rho}\Big)\Big] \preccurlyeq (1,1,1) \text{ and } \sup_{abc}\Big[\Theta_1\Big(\frac{\mathcal{S}((\mathfrak{A}_1)_{abc},(\mathfrak{S}_1)_{abc})}{\rho}\Big) \curlyvee \Theta_2\Big(\frac{\mathcal{S}((\mathfrak{A}_2)_{abc},(\mathfrak{S}_2)_{abc})}{\rho}\Big) \curlyvee \Theta_3\Big(\frac{\mathcal{S}((\mathfrak{A}_3)_{abc},(\mathfrak{S}_3)_{abc})}{\rho}\Big)\Big] \preccurlyeq (1,1,1)\Big\} =$$

$$\inf\Big\{(\rho,\rho,\rho) \succ (0,0,0): \sup_{abc}\Big[\Theta_1\Big(\frac{\mathcal{T}((\mathfrak{S}_1)_{abc},(\mathfrak{A}_1)_{abc})}{\rho}\Big) \curlyvee \Theta_2\Big(\frac{\mathcal{T}((\mathfrak{S}_2)_{abc},(\mathfrak{A}_2)_{abc})}{\rho}\Big) \curlyvee \Theta_3\Big(\frac{\mathcal{T}((\mathfrak{S}_3)_{abc},(\mathfrak{A}_3)_{abc})}{\rho}\Big)\Big] \preccurlyeq (1,1,1) \text{ and } \sup_{abc}\Big[\Theta_1\Big(\frac{\mathcal{S}((\mathfrak{S}_1)_{abc},(\mathfrak{A}_1)_{abc})}{\rho}\Big) \curlyvee \Theta_2\Big(\frac{\mathcal{S}((\mathfrak{S}_2)_{abc},(\mathfrak{A}_2)_{abc})}{\rho}\Big) \curlyvee \Theta_3\Big(\frac{\mathcal{S}((\mathfrak{S}_3)_{abc},(\mathfrak{A}_3)_{abc})}{\rho}\Big)\Big] \preccurlyeq (1,1,1)\Big\} = \bar{d}(\mathfrak{S},\mathfrak{A})_{\Theta}.$$

Therefore $\bar{d}(\mathfrak{A},\mathfrak{S})_{\Theta} = \bar{d}(\mathfrak{S},\mathfrak{A})_{\Theta}$.

iii) assume that $\rho_1,\rho_2 \succ 0$, then

$$\sup_{abc}\Big[\Theta_1\Big(\frac{\mathcal{T}((\mathfrak{A}_1)_{abc},(\mathfrak{R}_1)_{abc})}{\rho_1}\Big) \curlyvee \Theta_2\Big(\frac{\mathcal{T}((\mathfrak{A}_2)_{abc},(\mathfrak{R}_2)_{abc})}{\rho_1}\Big) \curlyvee \Theta_2\Big(\frac{\mathcal{T}((\mathfrak{A}_2)_{abc},(\mathfrak{R}_2)_{abc})}{\rho_1}\Big)\Big] \preccurlyeq (1,1,1),$$

and

$$\sup_{abc}\Big[\Theta_1\Big(\frac{\mathcal{T}((\mathfrak{R}_1)_{abc},(\mathfrak{S}_1)_{abc})}{\rho_2}\Big) \curlyvee \Theta_2\Big(\frac{\mathcal{T}((\mathfrak{R}_2)_{abc},(\mathfrak{S}_2)_{abc})}{\rho_2}\Big) \curlyvee \Theta_3\Big(\frac{\mathcal{T}((\mathfrak{R}_3)_{abc},(\mathfrak{S}_3)_{abc})}{\rho_2}\Big)\Big] \preccurlyeq (1,1,1).$$

Suppose that $\rho = \rho_1 + \rho_2$, using $\mathcal{T}'$s definition, we obtain

$$\mathcal{T}((\mathfrak{A}_1)_{abc},(\mathfrak{S}_1)_{abc}) = \sup_{0<\ltimes\leqslant 1}\mathcal{T}_{\ltimes}((\mathfrak{A}_1)^{\ltimes}_{abc},(\mathfrak{S}_1)^{\ltimes}_{abc}), \mathcal{T}((\mathfrak{A}_2)_{abc},(\mathfrak{S}_2)_{abc}) = \sup_{0<\ltimes\leqslant 1}\mathcal{T}_{\ltimes}((\mathfrak{A}_2)^{\ltimes}_{abc},(\mathfrak{S}_2)^{\ltimes}_{abc}), \mathcal{T}((\mathfrak{A}_3)_{abc},(3)_{abc}) = \sup_{0<\ltimes\leqslant 1}\mathcal{T}_{\ltimes}((\mathfrak{A}_3)^{\ltimes}_{abc},(\mathfrak{S}_3)^{\ltimes}_{abc})$$

and

$$\mathcal{T}_{\ltimes}((\mathfrak{A}_1)^{\ltimes}_{abc},(\mathfrak{S}_1)^{\ltimes}_{abc}) = \min\{|(\mathfrak{A}_1)^{\ltimes}_{abc1} - (\mathfrak{S}_1)^{\ltimes}_{abc1}|,|(\mathfrak{A}_1)^{\ltimes}_{abc2} - (\mathfrak{S}_1)^{\ltimes}_{abc2}|\}, \mathcal{T}_{\ltimes}((\mathfrak{A}_2)^{\ltimes}_{abc},(\mathfrak{S}_2)^{\ltimes}_{abc}) = \min\{|(\mathfrak{A}_2)^{\ltimes}_{abc1} - (\mathfrak{S}_2)^{\ltimes}_{abc1}|,|(\mathfrak{A}_2)^{\ltimes}_{abc2} - (\mathfrak{S}_2)^{\ltimes}_{abc2}|\}, \mathcal{T}_{\ltimes}((\mathfrak{A}_3)^{\ltimes}_{abc},(\mathfrak{S}_3)^{\ltimes}_{abc}) = \min\{|(\mathfrak{A}_3)^{\ltimes}_{abc1} - (\mathfrak{S}_3)^{\ltimes}_{abc1}|,|(\mathfrak{A}_3)^{\ltimes}_{abc2} - (\mathfrak{S}_3)^{\ltimes}_{abc2}|\}.$$

Using $\mathcal{T}$'s definition, we have

$$\mathcal{T}_{\ltimes}((\mathfrak{A}_1)^{\ltimes},(\mathfrak{S}_1)^{\ltimes}) \preccurlyeq \mathcal{T}_{\ltimes}((\mathfrak{A}_1)^{\ltimes},(\mathfrak{R}_1)^{\ltimes}) + \mathcal{T}_{\ltimes}((\mathfrak{R}_1)^{\ltimes},(\mathfrak{S}_1)^{\ltimes}), \mathcal{T}_{\ltimes}((\mathfrak{A}_2)^{\ltimes},(\mathfrak{S}_2)^{\ltimes}) \preccurlyeq \mathcal{T}_{\ltimes}((\mathfrak{A}_2)^{\ltimes},(\mathfrak{R}_2)^{\ltimes}) + \mathcal{T}_{\ltimes}((\mathfrak{R}_2)^{\ltimes},(\mathfrak{S}_2)^{\ltimes}), \mathcal{T}_{\ltimes}((\mathfrak{A}_3)^{\ltimes},(\mathfrak{S}_3)^{\ltimes}) \preccurlyeq \mathcal{T}_{\ltimes}((\mathfrak{A}_3)^{\ltimes},(\mathfrak{R}_3)^{\ltimes}) + \mathcal{T}_{\ltimes}((\mathfrak{R}_3)^{\ltimes},(\mathfrak{S}_3)^{\ltimes}), \forall \ltimes \in (0,1].$$

When we take the supremum of $\ltimes$, we obtain

$$\sup_{0<\ltimes\leqslant 1}\mathcal{T}_{\ltimes}((\mathfrak{A}_1)^{\ltimes},(\mathfrak{S}_1)^{\ltimes}) \preccurlyeq \sup_{0\leqslant\ltimes\leqslant 1}\mathcal{T}_{\ltimes}((\mathfrak{A}_1)^{\ltimes},(\mathfrak{R}_1)^{\ltimes}) + \sup_{0\leqslant\ltimes\leqslant 1}\mathcal{T}_{\ltimes}((\mathfrak{R}_1)^{\ltimes},(\mathfrak{S}_1)^{\ltimes}), \sup_{0<\ltimes\leqslant 1}\mathcal{T}_{\ltimes}((\mathfrak{A}_2)^{\ltimes},(\mathfrak{S}_2)^{\ltimes}) \preccurlyeq \sup_{0\leqslant\ltimes\leqslant 1}\mathcal{T}_{\ltimes}((\mathfrak{A}_2)^{\ltimes},(\mathfrak{R}_2)^{\ltimes}) + \sup_{0\leqslant\ltimes\leqslant 1}\mathcal{T}_{\ltimes}((\mathfrak{R}_2)^{\ltimes},(\mathfrak{S}_2)^{\ltimes}), \sup_{0<\ltimes\leqslant 1}\mathcal{T}_{\ltimes}((\mathfrak{A}_3)^{\ltimes},(\mathfrak{S}_3)^{\ltimes}) \preccurlyeq \sup_{0\leqslant\ltimes\leqslant 1}\mathcal{T}_{\ltimes}((\mathfrak{A}_3)^{\ltimes},(\mathfrak{R}_3)^{\ltimes}) + \sup_{0\leqslant\ltimes\leqslant 1}\mathcal{T}_{\ltimes}((\mathfrak{R}_3)^{\ltimes},(\mathfrak{S}_3)^{\ltimes}).$$

Moreover,

$$\mathcal{T}((\mathfrak{A}_1)_{abc},(\mathfrak{S}_1)_{abc}) \preccurlyeq \mathcal{T}((\mathfrak{A}_1)_{abc},(\mathfrak{R}_1)_{abc}) + \mathcal{T}((\mathfrak{R}_1)_{abc},(\mathfrak{S}_1)_{abc}), \mathcal{T}((\mathfrak{A}_2)_{abc},(\mathfrak{S}_2)_{abc}) \preccurlyeq \mathcal{T}((\mathfrak{A}_2)_{abc},(\mathfrak{R}_2)_{abc}) + \mathcal{T}((\mathfrak{R}_2)_{abc},(\mathfrak{S}_2)_{abc}), \mathcal{T}((\mathfrak{A}_3)_{abc},(\mathfrak{S}_3)_{abc}) \preccurlyeq \mathcal{T}((\mathfrak{A}_3)_{abc},(\mathfrak{R}_3)_{abc}) + \mathcal{T}((\mathfrak{R}_3)_{abc},(\mathfrak{S}_3)_{abc}).$$

Based on Θ's continuity, we determine that,

$$\sup_{abc}\Big[\Theta_1\Big(\frac{\mathcal{T}((\mathfrak{A}_1)_{abc},(\mathfrak{S}_1)_{abc})}{\rho}\Big) \curlyvee \Theta_2\Big(\frac{\mathcal{T}((\mathfrak{A}_2)_{abc},(\mathfrak{S}_2)_{abc})}{\rho}\Big) \curlyvee \Theta_3\Big(\frac{\mathcal{T}((\mathfrak{A}_3)_{abc},(\mathfrak{S}_3)_{abc})}{\rho}\Big)\Big]$$

$$\preccurlyeq \sup_{abc}\Big[\Theta_1\Big(\frac{\mathcal{T}((\mathfrak{A}_1)_{abc},(\mathfrak{R}_1)_{abc})}{\rho_1+\rho_2} + \frac{\mathcal{T}((\mathfrak{R}_1)_{abc},(\mathfrak{S}_1)_{abc})}{\rho_1+\rho_2}\Big) \curlyvee \Theta_2\Big(\frac{\mathcal{T}((\mathfrak{A}_2)_{abc},(\mathfrak{R}_2)_{abc})}{\rho_1+\rho_2} + \frac{\mathcal{T}((\mathfrak{R}_2)_{abc},(\mathfrak{S}_2)_{abc})}{\rho_1+\rho_2}\Big) \curlyvee \Theta_3\Big(\frac{\mathcal{T}((\mathfrak{A}_3)_{abc},(\mathfrak{R}_3)_{abc})}{\rho_1+\rho_2} + \frac{\mathcal{T}((\mathfrak{R}_3)_{abc},(\mathfrak{S}_3)_{abc})}{\rho_1+\rho_2}\Big)\Big]$$

$$\preccurlyeq \sup_{abc}\Big[\Theta_1\Big[\Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\Big(\frac{\mathcal{T}((\mathfrak{A}_1)_{abc},(\mathfrak{R}_1)_{abc})}{\rho_1}\Big) + \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)\Big(\frac{\mathcal{T}((\mathfrak{R}_1)_{abc},(\mathfrak{S}_1)_{abc})}{\rho_2}\Big)\Big] \curlyvee \Theta_2\Big[\Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\Big(\frac{\mathcal{T}((\mathfrak{A}_2)_{abc},(\mathfrak{R}_2)_{abc})}{\rho_1}\Big) + \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)\Big(\frac{\mathcal{T}((\mathfrak{R}_2)_{abc},(\mathfrak{S}_2)_{abc})}{\rho_2}\Big)\Big] \curlyvee \Theta_3\Big[\Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\Big(\frac{\mathcal{T}((\mathfrak{A}_3)_{abc},(\mathfrak{R}_3)_{abc})}{\rho_1}\Big) + \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)\Big(\frac{\mathcal{T}((\mathfrak{R}_3)_{abc},(\mathfrak{S}_3)_{abc})}{\rho_2}\Big)\Big]\Big]$$

$\preccurlyeq \sup_{abc}\left(\frac{\rho_1}{\rho_1+\rho_2}\right)\left[\Theta_1\left[\left(\frac{\mathcal{T}((\mathfrak{A}_1)_{abc},(\mathfrak{R}_1)_{abc})}{\rho_1}\right)\right] \curlyvee \Theta_2\left[\left(\frac{\mathcal{T}((\mathfrak{A}_2)_{abc},(\mathfrak{R}_2)_{abc})}{\rho_1}\right)\right] \curlyvee \Theta_3\left[\left(\frac{\mathcal{T}((\mathfrak{A}_3)_{abc},(\mathfrak{R}_3)_{abc})}{\rho_1}\right)\right]\right]$
$+ \sup_{abc}\left(\frac{\rho_2}{\rho_1+\rho_2}\right)\left[\Theta_1\left[\left(\frac{\mathcal{T}((\mathfrak{R}_1)_{abc},(\mathfrak{S}_1)_{abc})}{\rho_2}\right)\right] \curlyvee \Theta_2\left[\left(\frac{\mathcal{T}((\mathfrak{R}_2)_{abc},(\mathfrak{S}_2)_{abc})}{\rho_2}\right)\right] \curlyvee \Theta_3\left[\left(\frac{\mathcal{T}((\mathfrak{R}_3)_{abc},(\mathfrak{S}_3)_{abc})}{\rho_2}\right)\right]\right] \preccurlyeq (1,1,1).$

Given that ρ's are non-negative, then the infimum of these ρ's is introduced by

$\inf\left\{(\rho,\rho,\rho) \succ (0,0,0) : \sup_{abc}\left[\Theta_1\left(\frac{\mathcal{T}((\mathfrak{A}_1)_{abc},(\mathfrak{S}_1)_{abc})}{\rho}\right) \curlyvee \Theta_2\left(\frac{\mathcal{T}((\mathfrak{A}_2)_{abc},(\mathfrak{S}_2)_{abc})}{\rho}\right) \curlyvee \Theta_3\left(\frac{\mathcal{T}((\mathfrak{A}_3)_{abc},(\mathfrak{S}_3)_{abc})}{\rho}\right)\right] \preccurlyeq (1,1,1)\right\}$
$\preccurlyeq \inf\left\{(\rho_1,\rho_1,\rho_1) \succ (0,0,0) : \sup_{abc}\left[\Theta_1\left[\left(\frac{\mathcal{T}((\mathfrak{A}_1)_{abc},(\mathfrak{R}_1)_{abc})}{\rho_1}\right)\right] \curlyvee \Theta_2\left[\left(\frac{\mathcal{T}((\mathfrak{A}_2)_{abc},(\mathfrak{R}_2)_{abc})}{\rho_1}\right)\right] \curlyvee \Theta_3\left[\left(\frac{\mathcal{T}((\mathfrak{A}_3)_{abc},(\mathfrak{R}_3)_{abc})}{\rho_1}\right)\right]\right] \preccurlyeq (1,1,1)\right\}$
$+ \inf\left\{(\rho_2,\rho_2,\rho_2) \succ (0,0,0) : \sup_{abc}\left[\Theta_1\left[\left(\frac{\mathcal{T}((\mathfrak{R}_1)_{abc},(\mathfrak{S}_1)_{abc})}{\rho_2}\right)\right] \curlyvee \Theta_2\left[\left(\frac{\mathcal{T}((\mathfrak{R}_2)_{abc},(\mathfrak{S}_2)_{abc})}{\rho_2}\right)\right] \curlyvee \Theta_3\left[\left(\frac{\mathcal{T}((\mathfrak{R}_3)_{abc},(\mathfrak{S}_3)_{abc})}{\rho_2}\right)\right]\right] \preccurlyeq (1,1,1)\right\}.$

Following the same path, we eventually arrive at

$\inf\left\{(\rho,\rho,\rho) \succ (0,0,0) : \sup_{abc}\left[\Theta_1\left(\frac{\mathcal{S}((\mathfrak{A}_1)_{abc},(\mathfrak{S}_1)_{abc})}{\rho}\right) \curlyvee \Theta_2\left(\frac{\mathcal{S}((\mathfrak{A}_2)_{abc},(\mathfrak{S}_2)_{abc})}{\rho}\right) \curlyvee \Theta_3\left(\frac{\mathcal{S}((\mathfrak{A}_3)_{abc},(\mathfrak{S}_3)_{abc})}{\rho}\right)\right] \preccurlyeq (1,1,1)\right\}$
$\preccurlyeq \inf\left\{(\rho_1,\rho_1,\rho_1) \succ (0,0,0) : \sup_{abc}\left[\Theta_1\left[\left(\frac{\mathcal{S}((\mathfrak{A}_1)_{abc},(\mathfrak{R}_1)_{abc})}{\rho_1}\right)\right] \curlyvee \Theta_2\left[\left(\frac{\mathcal{S}((\mathfrak{A}_2)_{abc},(\mathfrak{R}_2)_{abc})}{\rho_1}\right)\right] \curlyvee \Theta_3\left[\left(\frac{\mathcal{S}((\mathfrak{A}_3)_{abc},(\mathfrak{R}_3)_{abc})}{\rho_1}\right)\right]\right] \preccurlyeq (1,1,1)\right\}$
$+ \inf\left\{(\rho_2,\rho_2,\rho_2) \succ (0,0,0) : \sup_{abc}\left[\Theta_1\left[\left(\frac{\mathcal{S}((\mathfrak{R}_1)_{abc},(\mathfrak{S}_1)_{abc})}{\rho_2}\right)\right] \curlyvee \Theta_2\left[\left(\frac{\mathcal{S}((\mathfrak{R}_2)_{abc},(\mathfrak{S}_2)_{abc})}{\rho_2}\right)\right] \curlyvee \Theta_3\left[\left(\frac{\mathcal{S}((\mathfrak{R}_3)_{abc},(\mathfrak{S}_3)_{abc})}{\rho_2}\right)\right]\right] \preccurlyeq (1,1,1)\right\}.$

Then,

$\inf\left\{(\rho,\rho,\rho) \succ (0,0,0) : \sup_{abc}\left[\Theta_1\left(\frac{\mathcal{T}((\mathfrak{A}_1)_{abc},(\mathfrak{S}_1)_{abc})}{\rho}\right) \curlyvee \Theta_2\left(\frac{\mathcal{T}((\mathfrak{A}_2)_{abc},(\mathfrak{S}_2)_{abc})}{\rho}\right) \curlyvee \Theta_3\left(\frac{\mathcal{T}((\mathfrak{A}_3)_{abc},(\mathfrak{S}_3)_{abc})}{\rho}\right)\right] \preccurlyeq (1,1,1) \text{ and } \sup_{abc}\left[\Theta_1\left(\frac{\mathcal{S}((\mathfrak{A}_1)_{abc},(\mathfrak{S}_1)_{abc})}{\rho}\right) \curlyvee \Theta_2\left(\frac{\mathcal{S}((\mathfrak{A}_2)_{abc},(\mathfrak{S}_2)_{abc})}{\rho}\right) \curlyvee \Theta_3\left(\frac{\mathcal{S}((\mathfrak{A}_3)_{abc},(\mathfrak{S}_3)_{abc})}{\rho}\right)\right] \preccurlyeq (1,1,1)\right\}$
$\preccurlyeq$
$\inf\left\{(\rho_1,\rho_1,\rho_1) \succ (0,0,0) : \sup_{abc}\left[\Theta_1\left[\left(\frac{\mathcal{T}((\mathfrak{A}_1)_{abc},(\mathfrak{R}_1)_{abc})}{\rho_1}\right)\right] \curlyvee \Theta_2\left[\left(\frac{\mathcal{T}((\mathfrak{A}_2)_{abc},(\mathfrak{R}_2)_{abc})}{\rho_1}\right)\right] \curlyvee \Theta_3\left[\left(\frac{\mathcal{T}((\mathfrak{A}_3)_{abc},(\mathfrak{R}_3)_{abc})}{\rho_1}\right)\right]\right] \preccurlyeq (1,1,1), \text{and } \sup_{abc}\left[\Theta_1\left[\left(\frac{\mathcal{S}((\mathfrak{A}_1)_{abc},(\mathfrak{R}_1)_{abc})}{\rho_1}\right)\right] \curlyvee \Theta_2\left[\left(\frac{\mathcal{S}((\mathfrak{A}_2)_{abc},(\mathfrak{R}_2)_{abc})}{\rho_1}\right)\right] \curlyvee \Theta_3\left[\left(\frac{\mathcal{S}((\mathfrak{A}_3)_{abc},(\mathfrak{R}_3)_{abc})}{\rho_1}\right)\right]\right] \preccurlyeq (1,1,1)\right\}$
$+ \inf\left\{(\rho_2,\rho_2,\rho_2) \succ (0,0,0) : \sup_{abc}\left[\Theta_1\left[\left(\frac{\mathcal{T}((\mathfrak{R}_1)_{abc},(\mathfrak{S}_1)_{abc})}{\rho_2}\right)\right] \curlyvee \Theta_2\left[\left(\frac{\mathcal{T}((\mathfrak{R}_2)_{abc},(\mathfrak{S}_2)_{abc})}{\rho_2}\right)\right] \curlyvee \Theta_3\left[\left(\frac{\mathcal{T}((\mathfrak{R}_3)_{abc},(\mathfrak{S}_3)_{abc})}{\rho_2}\right)\right]\right] \preccurlyeq (1,1,1) \text{ and } \sup_{abc}\left[\Theta_1\left[\left(\frac{\mathcal{T}((\mathfrak{R}_1)_{abc},(\mathfrak{S}_1)_{abc})}{\rho_2}\right)\right] \curlyvee \Theta_2\left[\left(\frac{\mathcal{T}((\mathfrak{R}_2)_{abc},(\mathfrak{S}_2)_{abc})}{\rho_2}\right)\right] \curlyvee \Theta_3\left[\left(\frac{\mathcal{T}((\mathfrak{R}_3)_{abc},(\mathfrak{S}_3)_{abc})}{\rho_2}\right)\right]\right] \preccurlyeq (1,1,1)\right\}.$

Therefore $\bar{d}(\mathfrak{A},\mathfrak{S})_\Theta \preccurlyeq \bar{d}(\mathfrak{A},\mathfrak{R})_\Theta + \bar{d}(\mathfrak{R},\mathfrak{S})_\Theta$.
Thus,
$(\ell_\infty)^3_{\mathbb{F}}(\Theta)$ is metric space.

Theorem 3.2:

Let $(\ell_\infty)^3_{\mathbb{F}}(\Theta)$ be a complete space under the metric:

$\bar{d}(\mathfrak{A},\mathfrak{S})_\Theta = \inf\left\{(\rho,\rho,\rho) \succ (0,0,0) : \sup_{abc}\left[\Theta_1\left(\frac{\mathcal{T}((\mathfrak{A}_1)_{abc},(\mathfrak{S}_1)_{abc})}{\rho}\right) \curlyvee \Theta_2\left(\frac{\mathcal{T}((\mathfrak{A}_2)_{abc},(\mathfrak{S}_2)_{abc})}{\rho}\right) \curlyvee \Theta_3\left(\frac{\mathcal{T}((\mathfrak{A}_3)_{abc},(\mathfrak{S}_3)_{abc})}{\rho}\right)\right] \preccurlyeq (1,1,1), \text{and } \sup_{abc}\left[\Theta_1\left(\frac{\mathcal{S}((\mathfrak{A}_1)_{abc},(\mathfrak{S}_1)_{abc})}{\rho}\right) \curlyvee \Theta_2\left(\frac{\mathcal{S}((\mathfrak{A}_2)_{abc},(\mathfrak{S}_2)_{abc})}{\rho}\right) \curlyvee \Theta_3\left(\frac{\mathcal{S}((\mathfrak{A}_3)_{abc},(\mathfrak{S}_3)_{abc})}{\rho}\right)\right] \preccurlyeq (1,1,1)\right\}, \forall\, \mathfrak{A} = (\mathfrak{A}_1,\mathfrak{A}_2,\mathfrak{A}_3), \mathfrak{S} = (\mathfrak{S}_1,\mathfrak{S}_2,\mathfrak{S}_3) \in (\ell_\infty)^3_{\mathbb{F}}(\Theta).$

Proof:

Assume that $((\mathfrak{R}_1)^{(ji\hbar)}), ((\mathfrak{R}_2)^{(ji\hbar)})$ and $((\mathfrak{R}_3)^{(ji\hbar)})$ are Cauchy triple sequence in $(\ell_\infty)^3_{\mathbb{F}}(\Theta) \ni (\mathfrak{R}_1)^{(ji\hbar)} =$

$\left((\mathfrak{R}_1)_{\mathfrak{uts}}^{(\mathscr{jih})}\right)_{\mathfrak{u,t,s}=1}^{\infty}$ and $(\mathfrak{R}_1)^{(\mathscr{jih})} =$ $\left((\mathfrak{R}_1)_{\mathfrak{uts}}^{(\mathscr{jih})}\right)_{\mathfrak{u,t,s}=1}^{\infty}$ and $(\mathfrak{R}_1)^{(\mathscr{jih})} = \left((\mathfrak{R}_1)_{\mathfrak{uts}}^{(\mathscr{jih})}\right)_{\mathfrak{u,t,s}=1}^{\infty}$.

Let $\varepsilon \succ 0$. For a fixed exist $x_0 \succ 0$, choose $\mathfrak{p} \succ 0 \ni$ $\left[\Theta_1\left(\frac{\mathfrak{p}x_0}{2}\right) \curlyvee \Theta_2\left(\frac{\mathfrak{p}x_0}{2}\right) \curlyvee \Theta_3\left(\frac{\mathfrak{p}x_0}{2}\right)\right] \succcurlyeq (1,1,1)$. ∃ a positive integer $n_0 = n_0(\varepsilon) \ni$

$$\bar{d}\begin{pmatrix}\left((\mathfrak{R}_1)^{(\mathscr{jih})}, (\mathfrak{R}_1)^{(\mathscr{fed})}\right), \left((\mathfrak{R}_2)^{(\mathscr{jih})}, (\mathfrak{R}_2)^{(\mathscr{fed})}\right), \\ \left((\mathfrak{R}_3)^{(\mathscr{jih})}, (\mathfrak{R}_3)^{(\mathscr{fed})}\right)\end{pmatrix}_{\mathbb{M}} \prec \left(\frac{\varepsilon}{\mathfrak{p}x_0}, \frac{\varepsilon}{\mathfrak{p}x_0}, \frac{\varepsilon}{\mathfrak{p}x_0}\right), \forall \mathscr{j}, \mathscr{i}, \mathscr{h}, \mathscr{f}, \mathscr{e}, \mathscr{d} \succcurlyeq n_0.$$

Using $\bar{d}_\Theta$'s definition, we obtain

$$\inf\Bigg\{(\rho,\rho,\rho) \succ (0,0,0) : \sup_{\mathfrak{abc}}\left[\Theta_1\left(\frac{\mathcal{T}\left((\mathfrak{R}_1)_{\mathfrak{abc}}^{(\mathscr{jih})},\ (\mathfrak{R}_1)_{\mathfrak{abc}}^{(\mathscr{fed})}\right)}{\rho}\right) \curlyvee \Theta_2\left(\frac{\mathcal{T}\left((\mathfrak{R}_2)_{\mathfrak{abc}}^{(\mathscr{jih})},\ (\mathfrak{R}_2)_{\mathfrak{abc}}^{(\mathscr{fed})}\right)}{\rho}\right) \curlyvee \Theta_3\left(\frac{\mathcal{T}\left((\mathfrak{R}_3)_{\mathfrak{abc}}^{(\mathscr{jih})},\ (\mathfrak{R}_3)_{\mathfrak{abc}}^{(\mathscr{fed})}\right)}{\rho}\right)\right] \preccurlyeq (1,1,1) \text{ and } \sup_{\mathfrak{abc}}\left[\Theta_1\left(\frac{\mathcal{S}\left((\mathfrak{R}_1)_{\mathfrak{abc}}^{(\mathscr{jih})},\ (\mathfrak{R}_1)_{\mathfrak{abc}}^{(\mathscr{fed})}\right)}{\rho}\right) \curlyvee \Theta_2\left(\frac{\mathcal{S}\left((\mathfrak{R}_2)_{\mathfrak{abc}}^{(\mathscr{jih})},\ (\mathfrak{R}_2)_{\mathfrak{abc}}^{(\mathscr{fed})}\right)}{\rho}\right) \curlyvee \Theta_3\left(\frac{\mathcal{S}\left((\mathfrak{R}_3)_{\mathfrak{abc}}^{(\mathscr{jih})},\ (\mathfrak{R}_3)_{\mathfrak{abc}}^{(\mathscr{fed})}\right)}{\rho}\right)\right] \preccurlyeq (1,1,1)\Bigg\} \prec (\varepsilon,\varepsilon,\varepsilon), \forall \mathscr{j},\mathscr{i},\mathscr{h},\mathscr{f},\mathscr{e},\mathscr{d} \succcurlyeq n_0 \ldots\ldots\ldots (1),$$

which leads to

$$\sup_{\mathfrak{abc}}\left[\Theta_1\left(\frac{\mathcal{T}\left((\mathfrak{R}_1)_{\mathfrak{abc}}^{(\mathscr{jih})},\ (\mathfrak{R}_1)_{\mathfrak{abc}}^{(\mathscr{fed})}\right)}{\rho}\right) \curlyvee \Theta_2\left(\frac{\mathcal{T}\left((\mathfrak{R}_2)_{\mathfrak{abc}}^{(\mathscr{jih})},\ (\mathfrak{R}_2)_{\mathfrak{abc}}^{(\mathscr{fed})}\right)}{\rho}\right) \curlyvee \Theta_3\left(\frac{\mathcal{T}\left((\mathfrak{R}_3)_{\mathfrak{abc}}^{(\mathscr{jih})},\ (\mathfrak{R}_3)_{\mathfrak{abc}}^{(\mathscr{fed})}\right)}{\rho}\right)\right] \preccurlyeq (1,1,1) \ \ldots\ldots (2) .$$

$$\sup_{\mathfrak{abc}}\left[\Theta_1\left(\frac{\mathcal{S}\left((\mathfrak{R}_1)_{\mathfrak{abc}}^{(\mathscr{jih})},\ (\mathfrak{R}_1)_{\mathfrak{abc}}^{(\mathscr{fed})}\right)}{\rho}\right) \curlyvee \Theta_2\left(\frac{\mathcal{S}\left((\mathfrak{R}_2)_{\mathfrak{abc}}^{(\mathscr{jih})},\ (\mathfrak{R}_2)_{\mathfrak{abc}}^{(\mathscr{fed})}\right)}{\rho}\right) \curlyvee \Theta_3\left(\frac{\mathcal{S}\left((\mathfrak{R}_3)_{\mathfrak{abc}}^{(\mathscr{jih})},\ (\mathfrak{R}_3)_{\mathfrak{abc}}^{(\mathscr{fed})}\right)}{\rho}\right)\right] \preccurlyeq (1,1,1) \ldots\ldots (3) .$$

From (2), we have

$$\left[\Theta_1\left(\frac{\mathcal{T}\left((\mathfrak{R}_1)_{\mathfrak{abc}}^{(\mathscr{jih})},\ (\mathfrak{R}_1)_{\mathfrak{abc}}^{(\mathscr{fed})}\right)}{\bar{d}\left((\mathfrak{R}_1)^{(\mathscr{jih})},(\mathfrak{R}_1)^{(\mathscr{fed})}\right)}\right) \curlyvee \Theta_2\left(\frac{\mathcal{T}\left((\mathfrak{R}_2)_{\mathfrak{abc}}^{(\mathscr{jih})},\ (\mathfrak{R}_2)_{\mathfrak{abc}}^{(\mathscr{fed})}\right)}{\bar{d}\left((\mathfrak{R}_2)^{(\mathscr{jih})},(\mathfrak{R}_2)^{(\mathscr{fed})}\right)}\right) \curlyvee \Theta_3\left(\frac{\mathcal{T}\left((\mathfrak{R}_3)_{\mathfrak{abc}}^{(\mathscr{jih})},\ (\mathfrak{R}_3)_{\mathfrak{abc}}^{(\mathscr{fed})}\right)}{\bar{d}\left((\mathfrak{R}_3)^{(\mathscr{jih})},(\mathfrak{R}_3)^{(\mathscr{fed})}\right)}\right)\right] \preccurlyeq (1,1,1) \preccurlyeq \left[\Theta_1\left(\frac{\mathfrak{p}x_0}{2}\right) \curlyvee \Theta_2\left(\frac{\mathfrak{p}x_0}{2}\right) \curlyvee \Theta_3\left(\frac{\mathfrak{p}x_0}{2}\right)\right].$$

By Θ′s continuity, we determine that,

$$\mathcal{T}\begin{pmatrix}\left((\mathfrak{R}_1)_{\mathfrak{abc}}^{(\mathscr{jih})}, (\mathfrak{R}_1)_{\mathfrak{abc}}^{(\mathscr{fed})}\right), \left((\mathfrak{R}_2)_{\mathfrak{abc}}^{(\mathscr{jih})}, (\mathfrak{R}_2)_{\mathfrak{abc}}^{(\mathscr{fed})}\right), \\ \left((3)_{\mathfrak{abc}}^{(\mathscr{jih})}, (\mathfrak{R}_3)_{\mathfrak{abc}}^{(\mathscr{fed})}\right)\end{pmatrix} \preccurlyeq \left(\frac{\mathfrak{p}x_0}{2}, \frac{\mathfrak{p}x_0}{2}, \frac{\mathfrak{p}x_0}{2}\right) \cdot \left(\frac{\varepsilon}{\mathfrak{p}x_0}, \frac{\varepsilon}{\mathfrak{p}x_0}, \frac{\varepsilon}{\mathfrak{p}x_0}\right) = \left(\frac{\varepsilon}{2}, \frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right).$$

According to the completeness property of $\mathbb{R}(\mathbb{I})$, $\left((\mathfrak{R}_1)_{\mathfrak{abc}}^{(\mathscr{jih})}\right), \left((\mathfrak{R}_2)_{\mathfrak{abc}}^{(\mathscr{jih})}\right), \left((\mathfrak{R}_3)_{\mathfrak{abc}}^{(\mathscr{jih})}\right)$ is convergent in $\mathbb{R}(\mathbb{I})$, since it is a Cauchy triple sequence in $\mathbb{R}(\mathbb{I})$.

Let $\lim_{\mathscr{jih}} (\mathfrak{R}_1)_{\mathfrak{abc}}^{(\mathscr{jih})} = (\mathfrak{R}_1)_{\mathfrak{abc}}$, $\lim_{\mathscr{jih}} (\mathfrak{R}_2)_{\mathfrak{abc}}^{(\mathscr{jih})} = (\mathfrak{R}_2)_{\mathfrak{abc}}$, $\lim_{\mathscr{jih}} (\mathfrak{R}_3)_{\mathfrak{abc}}^{(\mathscr{jih})} = (\mathfrak{R}_3)_{\mathfrak{abc}}$, $\forall\, \mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{e} \in \mathbb{N}$.

We must demonstrate that, $\lim_{\mathscr{jih}} (\mathfrak{R}_1)^{(\mathscr{jih})} = \mathfrak{R}_1$, $\lim_{\mathscr{jih}} (\mathfrak{R}_2)^{(\mathscr{jih})} = \mathfrak{R}_2$, $\lim_{\mathscr{jih}} (\mathfrak{R}_3)^{(\mathscr{jih})} = \mathfrak{R}_3$, $\forall\, \mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3 \in (\ell_\infty)_{\mathbb{F}}^3(\Theta)$

Θ being a continuous, taking $\mathscr{f}, \mathscr{e}, \mathscr{d} \to \infty$ and fixing $\mathscr{j}, \mathscr{i}, \mathscr{h}$. From (2), we obtain

$$\sup_{\mathfrak{abc}}\left[\Theta_1\left(\frac{\mathcal{T}\left((\mathfrak{R}_1)_{\mathfrak{abc}}^{(\mathscr{jih})}, (\mathfrak{R}_1)_{\mathfrak{abc}}\right)}{\rho}\right) \curlyvee \Theta_2\left(\frac{\mathcal{T}\left((\mathfrak{R}_2)_{\mathfrak{abc}}^{(\mathscr{jih})}, (\mathfrak{R}_2)_{\mathfrak{abc}}\right)}{\rho}\right) \curlyvee \Theta_3\left(\frac{\mathcal{T}\left((\mathfrak{R}_3)_{\mathfrak{abc}}^{(\mathscr{jih})}, (\mathfrak{R}_3)_{\mathfrak{abc}}\right)}{\rho}\right)\right] \preccurlyeq (1,1,1), \text{for some } \rho \succ 0, \forall \mathscr{j}, \mathscr{i}, \mathscr{h} \succcurlyeq n_0.$$

Continuing in the same manner, from (3), we have

$$\sup_{\mathfrak{abc}}\left[\Theta_1\left(\frac{\mathcal{S}\left((\mathfrak{R}_1)_{\mathfrak{abc}}^{(\mathscr{jih})}, (\mathfrak{R}_1)_{\mathfrak{abc}}\right)}{\rho}\right) \curlyvee \Theta_2\left(\frac{\mathcal{S}\left((\mathfrak{R}_2)_{\mathfrak{abc}}^{(\mathscr{jih})}, (\mathfrak{R}_2)_{\mathfrak{abc}}\right)}{\rho}\right) \curlyvee \Theta_3\left(\frac{\mathcal{S}\left((\mathfrak{R}_3)_{\mathfrak{abc}}^{(\mathscr{jih})}, (\mathfrak{R}_3)_{\mathfrak{abc}}\right)}{\rho}\right)\right] \preccurlyeq (1,1,1), \text{for some } \rho \succ 0, \forall \mathscr{j}, \mathscr{i}, \mathscr{h} \succcurlyeq n_0.$$

Now, taking the infimum for ρ′s, from (1), we obtain

$$\inf\Bigg\{(\rho,\rho,\rho) \succ (0,0,0) : \sup_{\mathfrak{abc}}\left[\Theta_1\left(\frac{\mathcal{T}\left((\mathfrak{R}_1)_{\mathfrak{abc}}^{(\mathscr{jih})}, (\mathfrak{R}_1)_{\mathfrak{abc}}\right)}{\rho}\right) \curlyvee \Theta_2\left(\frac{\mathcal{T}\left((\mathfrak{R}_2)_{\mathfrak{abc}}^{(\mathscr{jih})}, (\mathfrak{R}_2)_{\mathfrak{abc}}\right)}{\rho}\right) \curlyvee \Theta_3\left(\frac{\mathcal{T}\left((\mathfrak{R}_3)_{\mathfrak{abc}}^{(\mathscr{jih})}, (\mathfrak{R}_3)_{\mathfrak{abc}}\right)}{\rho}\right)\right] \preccurlyeq (1,1,1), \text{and } \sup_{\mathfrak{abc}}\left[\Theta_1\left(\frac{\mathcal{S}\left((\mathfrak{R}_1)_{\mathfrak{abc}}^{(\mathscr{jih})}, (\mathfrak{R}_1)_{\mathfrak{abc}}\right)}{\rho}\right) \curlyvee \Theta_2\left(\frac{\mathcal{S}\left((\mathfrak{R}_2)_{\mathfrak{abc}}^{(\mathscr{jih})}, (\mathfrak{R}_2)_{\mathfrak{abc}}\right)}{\rho}\right) \curlyvee \Theta_3\left(\frac{\mathcal{S}\left((\mathfrak{R}_3)_{\mathfrak{abc}}^{(\mathscr{jih})}, (\mathfrak{R}_3)_{\mathfrak{abc}}\right)}{\rho}\right)\right] \preccurlyeq (1,1,1)\Bigg\} \prec (\varepsilon,\varepsilon,\varepsilon), \forall \mathscr{j}, \mathscr{i}, \mathscr{h} \succcurlyeq n_0,$$

which tends to

$$\bar{d}\left(\left((\mathfrak{R}_1)^{(\mathscr{jih})}, \mathfrak{R}_1\right), \left((\mathfrak{R}_2)^{(\mathscr{jih})}, \mathfrak{R}_2\right), \left((\mathfrak{R}_3)^{(\mathscr{jih})}, \mathfrak{R}_3\right)\right)_\Theta \prec (\varepsilon,\varepsilon,\varepsilon), \forall \mathscr{j}, \mathscr{i}, \mathscr{h} \succcurlyeq n_0 \Longrightarrow \lim_{\mathscr{jih}} (\mathfrak{R}_1)^{(\mathscr{jih})} = \mathfrak{R}_1, \lim_{\mathscr{jih}} (\mathfrak{R}_2)^{(\mathscr{jih})} = \mathfrak{R}_2, \lim_{\mathscr{jih}} (\mathfrak{R}_3)^{(\mathscr{jih})} = \mathfrak{R}_3$$

Now, It' prove that $\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3 \in (\ell_\infty)_{\mathbb{F}}^3(\Theta)$.

Taking into account that,

$$\bar{d}\left((\mathfrak{R}_1, 0), (\mathfrak{R}_2, 0), (\mathfrak{R}_3, 0)\right)_\Theta \preccurlyeq \bar{d}\left(\left(\mathfrak{R}_1, (\mathfrak{R}_1)^{(\mathscr{jih})}\right), \left(\mathfrak{R}_2, (\mathfrak{R}_2)^{(\mathscr{jih})}\right), \left(\mathfrak{R}_3, (\mathfrak{R}_3)^{(\mathscr{jih})}\right)\right)_\Theta +$$

$\bar{d}\left(\left((\mathfrak{R}_1)^{(jih)},0\right),\left((\mathfrak{R}_1)^{(jih)},0\right),\left((\mathfrak{R}_1)^{(jih)},0\right)\right)_\Theta \prec (\varepsilon,\varepsilon,\varepsilon)+(\Theta_1,\Theta_2,\Theta_3), \forall j,i,h \succcurlyeq n_0(\varepsilon)$,

We conclude that $\bar{d}\left((\mathfrak{R}_1,0),(\mathfrak{R}_2,0),(\mathfrak{R}_3,0)\right)_\Theta$ is finite.

Therefore $\mathfrak{R}_1,\mathfrak{R}_2,\mathfrak{R}_3 \in (\ell_\infty)^3_{\mathbb{F}}(\Theta)$.

Thus ,

$(\ell_\infty)^3_{\mathbb{F}}(\Theta)$ is complete .

Theorem3.3:

$(\ell_\infty)^3_{\mathbb{F}}(\Theta)$ is solid.

Proof:

Suppose that $(\mathfrak{M}_{abc}) \in (\ell_\infty)^3_{\mathbb{F}}(\Theta)$. Then we have $\sup_{abc}\left[\Theta_1\left(\frac{\mathcal{T}((\mathfrak{M}_1)_{abc},\bar{0})}{\rho}\right) \curlyvee \Theta_2\left(\frac{\mathcal{T}((\mathfrak{M}_2)_{abc},\bar{0})}{\rho}\right) \curlyvee \Theta_3\left(\frac{\mathcal{T}((\mathfrak{M}_3)_{abc},\bar{0})}{\rho}\right)\right] \prec \infty$ and $\sup_{abc}\left[\Theta_1\left(\frac{\mathcal{S}((\mathfrak{M}_1)_{abc},\bar{0})}{\rho}\right) \curlyvee \Theta_2\left(\frac{\mathcal{S}((\mathfrak{M}_2)_{abc},\bar{0})}{\rho}\right) \curlyvee \Theta_3\left(\frac{\mathcal{S}((\mathfrak{M}_3)_{abc},\bar{0})}{\rho}\right)\right] \prec \infty$, for some $\rho \succ 0$.

Suppose $(\mathfrak{N}_{abc})$ is a sequence of fuzzy numbers with,

$[d((\mathfrak{N}_1)_{abc},\bar{0})]_\ltimes = [\mathcal{T}_\ltimes((\mathfrak{N}_1)^\ltimes_{abc},0), \mathcal{S}_\ltimes((\mathfrak{N}_1)^\ltimes_{abc},0)]$
and
$[d((\mathfrak{N}_2)_{abc},\bar{0})]_\ltimes = [\mathcal{T}_\ltimes((\mathfrak{N}_2)^\ltimes_{abc},0), \mathcal{S}_\ltimes((\mathfrak{N}_2)^\ltimes_{abc},0)]$
and
$[d((\mathfrak{N}_3)_{abc},\bar{0})]_\ltimes = [\mathcal{T}_\ltimes((\mathfrak{N}_1)^\ltimes_{abc},0), \mathcal{S}_\ltimes((\mathfrak{N}_1)^\ltimes_{abc},0)], \forall\, 0 \prec \ltimes \preccurlyeq 1$

Such that ,

$\mathcal{T}[((\mathfrak{N}_1)_{abc},\bar{0})] \preccurlyeq \mathcal{T}[((\mathfrak{M}_1)_{abc},\bar{0})]$ and
$\mathcal{T}[(\mathfrak{N}_2)_{abc},\bar{0}], \preccurlyeq \mathcal{T}[(\mathfrak{M}_2)_{abc},\bar{0}]$ and
$\mathcal{T}[(\mathfrak{N}_3)_{abc},\bar{0}], \preccurlyeq \mathcal{T}[(\mathfrak{M}_3)_{abc},\bar{0}]$
and
$\mathcal{S}[((\mathfrak{N}_1)_{abc},\bar{0})] \preccurlyeq \mathcal{S}[((\mathfrak{M}_1)_{abc},\bar{0})]$ and
$\mathcal{S}[(\mathfrak{N}_2)_{abc},\bar{0}], \preccurlyeq \mathcal{S}[(\mathfrak{M}_2)_{abc},\bar{0}]$ and
$\mathcal{S}[(\mathfrak{N}_3)_{abc},\bar{0}], \preccurlyeq \mathcal{S}[(\mathfrak{M}_3)_{abc},\bar{0}]$.

Since Θ is a continuous and not diminishing, we get, for some $\rho \succ 0$,

$$\left[\Theta_1\left(\frac{\mathcal{T}((\mathfrak{N}_1)_{abc},\bar{0})}{\rho}\right) \curlyvee \Theta_2\left(\frac{\mathcal{T}((\mathfrak{N}_2)_{abc},\bar{0})}{\rho}\right) \curlyvee \Theta_3\left(\frac{\mathcal{T}((\mathfrak{N}_3)_{abc},\bar{0})}{\rho}\right)\right] \preccurlyeq \left[\Theta_1\left(\frac{\mathcal{T}((\mathfrak{M}_1)_{abc},\bar{0})}{\rho}\right) \curlyvee \Theta_2\left(\frac{\mathcal{T}((\mathfrak{M}_2)_{abc},\bar{0})}{\rho}\right) \curlyvee \Theta_3\left(\frac{\mathcal{T}((\mathfrak{M}_3)_{abc},\bar{0})}{\rho}\right)\right],$$

and

$$\left[\Theta_1\left(\frac{\mathcal{S}((\mathfrak{N}_1)_{abc},\bar{0})}{\rho}\right) \curlyvee \Theta_2\left(\frac{\mathcal{S}((\mathfrak{N}_2)_{abc},\bar{0})}{\rho}\right) \curlyvee \Theta_3\left(\frac{\mathcal{S}((\mathfrak{N}_3)_{abc},\bar{0})}{\rho}\right)\right] \preccurlyeq \left[\Theta_1\left(\frac{\mathcal{S}((\mathfrak{M}_1)_{abc},\bar{0})}{\rho}\right) \curlyvee \Theta_2\left(\frac{\mathcal{S}((\mathfrak{M}_2)_{abc},\bar{0})}{\rho}\right) \curlyvee \Theta_3\left(\frac{\mathcal{S}((\mathfrak{M}_3)_{abc},\bar{0})}{\rho}\right)\right].$$

In addition,

$$\sup_{abc}\left[\Theta_1\left(\frac{\mathcal{T}((\mathfrak{N}_1)_{abc},\bar{0})}{\rho}\right) \curlyvee \Theta_2\left(\frac{\mathcal{T}((\mathfrak{N}_2)_{abc},\bar{0})}{\rho}\right) \curlyvee \Theta_3\left(\frac{\mathcal{T}((\mathfrak{N}_3)_{abc},\bar{0})}{\rho}\right)\right] \preccurlyeq \sup_{abc}\left[\Theta_1\left(\frac{\mathcal{T}((\mathfrak{M}_1)_{abc},\bar{0})}{\rho}\right) \curlyvee \Theta_2\left(\frac{\mathcal{T}((\mathfrak{M}_2)_{abc},\bar{0})}{\rho}\right) \curlyvee \Theta_3\left(\frac{\mathcal{T}((\mathfrak{M}_3)_{abc},\bar{0})}{\rho}\right)\right] \prec \infty$$, for some $\rho \succ 0$.

$$\sup_{abc}\left[\Theta_1\left(\frac{\mathcal{S}((\mathfrak{N}_1)_{abc},\bar{0})}{\rho}\right) \curlyvee \Theta_2\left(\frac{\mathcal{S}((\mathfrak{N}_2)_{abc},\bar{0})}{\rho}\right) \curlyvee \Theta_3\left(\frac{\mathcal{S}((\mathfrak{N}_3)_{abc},\bar{0})}{\rho}\right)\right] \preccurlyeq \sup_{abc}\left[\Theta_1\left(\frac{\mathcal{S}((\mathfrak{M}_1)_{abc},\bar{0})}{\rho}\right) \curlyvee \Theta_2\left(\frac{\mathcal{S}((\mathfrak{M}_2)_{abc},\bar{0})}{\rho}\right) \curlyvee \Theta_3\left(\frac{\mathcal{S}((\mathfrak{M}_3)_{abc},\bar{0})}{\rho}\right)\right] \prec \infty$$, for some $\rho \succ 0$.

Moreover , we have

$\sup_{abc}\left[\Theta_1\left(\frac{\mathcal{T}((\mathfrak{N}_1)_{abc},\bar{0})}{\rho}\right) \curlyvee \Theta_2\left(\frac{\mathcal{T}((\mathfrak{N}_2)_{abc},\bar{0})}{\rho}\right) \curlyvee \Theta_3\left(\frac{\mathcal{T}((\mathfrak{N}_3)_{abc},\bar{0})}{\rho}\right)\right] \prec \infty$ and $\sup_{abc}\left[\Theta_1\left(\frac{\mathcal{S}((\mathfrak{N}_1)_{abc},\bar{0})}{\rho}\right) \curlyvee \Theta_2\left(\frac{\mathcal{S}((\mathfrak{N}_2)_{abc},\bar{0})}{\rho}\right) \curlyvee \Theta_3\left(\frac{\mathcal{S}((\mathfrak{N}_3)_{abc},\bar{0})}{\rho}\right)\right] \prec \infty$.

Therefore $(\mathfrak{N}_{abc}) \in (\ell_\infty)^3_{\mathbb{F}}(\Theta)$.

Thus ,

$(\ell_\infty)^3_{\mathbb{F}}(\Theta)$ is solid .

Theorem 3.4:

$(\ell_\infty)^3_{\mathbb{F}}(\Theta)$ is symmetric.

Proof:

Assume $(\mathfrak{M}_{abc}) \in (\ell_\infty)^3_{\mathbb{F}}(\Theta)$ and $(\mathfrak{N}_{abc})$ is a reorganized of $(\mathfrak{M}_{abc}) \ni \mathfrak{M}_{abc} = \mathfrak{N}_{qpn_{abc}}, \forall\, a,b,c \in \mathbb{N}$.

Then, we have

$\mathcal{T}\left(\left((\mathfrak{N}_1)_{qpn_{abc}},\bar{0}\right),\left((\mathfrak{N}_2)_{qpn_{abc}},\bar{0}\right),\left((\mathfrak{N}_3)_{qpn_{abc}},\bar{0}\right)\right) = \mathcal{T}\left(((\mathfrak{M}_1)_{abc},\bar{0}),((\mathfrak{M}_2)_{abc},\bar{0}),((\mathfrak{M}_2)_{abc},\bar{0})\right)$,

and

$\mathcal{S}\left(\left((\mathfrak{N}_1)_{qpn_{abc}},\bar{0}\right),\left((\mathfrak{N}_2)_{qpn_{abc}},\bar{0}\right),\left((\mathfrak{N}_3)_{qpn_{abc}},\bar{0}\right)\right) = \mathcal{S}\left(((\mathfrak{M}_1)_{abc},\bar{0}),((\mathfrak{M}_2)_{abc},\bar{0}),((\mathfrak{M}_2)_{abc},\bar{0})\right)$.

Based on Θ's continuity, we determine that,

$$\sup_{abc}\left[\Theta_1\left(\frac{\mathcal{T}\left((\mathfrak{N}_1)_{qpn_{abc}},\bar{0}\right)}{\rho}\right) \curlyvee \Theta_2\left(\frac{\mathcal{T}\left((\mathfrak{N}_2)_{qpn_{abc}},\bar{0}\right)}{\rho}\right) \curlyvee \Theta_3\left(\frac{\mathcal{T}\left((\mathfrak{N}_3)_{qpn_{abc}},\bar{0}\right)}{\rho}\right)\right] = \sup_{abc}\left[\Theta_1\left(\frac{\mathcal{T}((\mathfrak{M}_1)_{abc},\bar{0})}{\rho}\right) \curlyvee \Theta_2\left(\frac{\mathcal{T}((\mathfrak{M}_2)_{abc},\bar{0})}{\rho}\right) \curlyvee \Theta_3\left(\frac{\mathcal{T}((\mathfrak{M}_3)_{abc},\bar{0})}{\rho}\right)\right]$$, for some $\rho \succ 0$,

and

$$\sup_{abc}\left[\Theta_1\left(\frac{\mathcal{S}\left((\mathfrak{N}_1)_{qpn_{abc}},\bar{0}\right)}{\rho}\right) \curlyvee \Theta_2\left(\frac{\mathcal{S}\left((\mathfrak{N}_2)_{qpn_{abc}},\bar{0}\right)}{\rho}\right) \curlyvee \Theta_3\left(\frac{\mathcal{S}\left((\mathfrak{N}_3)_{qpn_{abc}},\bar{0}\right)}{\rho}\right)\right] = \sup_{abc}\left[\Theta_1\left(\frac{\mathcal{S}((\mathfrak{M}_1)_{abc},\bar{0})}{\rho}\right) \curlyvee \Theta_2\left(\frac{\mathcal{S}((\mathfrak{M}_2)_{abc},\bar{0})}{\rho}\right) \curlyvee \Theta_3\left(\frac{\mathcal{S}((\mathfrak{M}_3)_{abc},\bar{0})}{\rho}\right)\right]$$, for some $\rho \succ 0$.

This means that ,

$\sup_{abc}\left[\Theta_1\left(\frac{\mathcal{T}\left((\mathfrak{N}_1)_{qpn_{abc}},\bar{0}\right)}{\rho}\right) \curlyvee \Theta_2\left(\frac{\mathcal{T}\left((\mathfrak{N}_2)_{qpn_{abc}},\bar{0}\right)}{\rho}\right) \curlyvee \Theta_3\left(\frac{\mathcal{T}\left((\mathfrak{N}_3)_{qpn_{abc}},\bar{0}\right)}{\rho}\right)\right] \prec (\infty,\infty,\infty)$ and $\sup_{abc}\left[\Theta_1\left(\frac{\mathcal{S}\left((\mathfrak{N}_1)_{qpn_{abc}},\bar{0}\right)}{\rho}\right) \curlyvee\right.$

$$\Theta_2\left(\frac{\mathcal{S}\left((\mathfrak{N}_2)_{\mathfrak{qpn}_{\mathfrak{abc}}},\bar{0}\right)}{\rho}\right) \curlyvee \Theta_3\left(\frac{\mathcal{S}\left((\mathfrak{N}_3)_{\mathfrak{qpn}_{\mathfrak{abc}}},\bar{0}\right)}{\rho}\right)\Bigg] \prec (\infty,\infty,\infty),$$

for some $\rho \succ 0$.

Therefore $(\mathfrak{N}_{\mathfrak{abc}}) \in (\ell_\infty)^3_{\mathbb{F}}(\Theta)$.

Thus ,

$(\ell_\infty)^3_{\mathbb{F}}(\Theta)$ is symmetric .

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Arabic Abstract

سوف نقدم دالة اوليسز المطلقة الناقصة البسيطة الثلاثية في هذا البحث , والمحددة بواسطة فضاءات المتتابعات الثلاثية مع المترية الضبابية , وكذلك سوف نناقش بعض الخواص , مثلا الفضاء $(\ell_\infty)^3_{\mathbb{F}}(\Theta)$ هو فضاء متناظر , فضاء صلب , فضاء كامل .