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A New Extended Inverse Exponential Distribution with Medical and Engineering Applications

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P A P E R I N F O

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A B S T R A C T

A new probability distribution named Truncated Rayleigh Odd Weibull Inverse Exponential distribution that extends the traditional inverse exponential distribution is proposed. The essential statistical properties including moments, quantile function, linear representation, measures of reliability, entropies, and reliability stress strength model are derived. The unknown three parameters are estimated with the method of maximum likelihood and a simulation study is introduced to examine the accuracy of the estimates. Two applications based on real-life datasets - medical and engineering - are considered. Due to its flexible features, the new extended distribution is preferable to number of well-known comparable models.

1. INTRODUCTION

Statistical data modeling is a crucial part of statistics that has drawn the attention of many researchers. A suitable statistical model is necessary for the accurate actualization of the data when modeling real-life data in various fields such as economics, reliability analysis, engineering, environment, biological investigations, and medical sciences. However, there are still issues when real-life data does not fit any of the conventional probability models. Indeed, statistical and applied researchers have expressed a strong interest in developing new extended probability distributions that are more adaptable to data modeling. The literature describes numerous methods for extending well-known distributions. One of the most prevalent approaches is to think about distribution generators [1,2].

For the generator (G) approach adding parameter (s) to the well-known distributions may introduce new modified/or extended distributions with high flexibility in data-driven modeling of real-life phenomena. In the literature, many generator families of probability distributions with several desirable properties have been proposed. A summary of the varied and useful proposed families includes the Marshall–Olkin-G [3], beta-G [4], Kumaraswamy-G [5], gamma-G [6], exponentiated generalized-G [7], logistic-G [8], Weibull-G [9], truncated Fréchet-G [10], Gompertz-G [11], generalized inverse Weibull-G [12], truncated general-G [13], exponentiated truncated inverse Weibull-G [14], and recently truncated Rayleigh odd Weibull-G [15] to the bounded interval [0,1]. For more families and details, the interested reader may refer to [14] and [15]. For lifetime data analysis, the exponential distribution is one of the most often used distributions related to the scale family of distributions due to its simplicity and mathematical viability. However, in real life, rarely come across engineering systems that have a consistent danger rate throughout their lifetime. As a result, it appears reasonable to assume hazard rate as a function of time which led to the development of an alternative modified/extended model for lifetime data analysis [16].

The inverse exponential (IE) distribution is an alternative modified version of the traditional exponential distribution proposed by [17]. But this modified version also has a limitation which is its inability to effectively represent various skewed datasets. The goal of this paper is centered around introducing a new version of the IE distribution based on employing the truncated Rayleigh odd Weibull-G ($\mathrm{TROW}-\mathrm{G}$) family. The new extended version is called truncated Rayleigh odd Weibull inverse exponential (TROWIE) to the bounded interval [0,1].

For $x > 0$ and $\bar{G}(x;\omega) = 1 - G(x;\omega)$, suppose $G(x;\omega)$ and $g(x;\omega) = dG(x;\omega)/dx$ are the cumulative distribution function (CDF) and probability density function (PDF) of a baseline distribution with parameter vector ω, then the CDF of $\mathrm{TROW}-\mathrm{G}$ family is [15]

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$$F(x) = \frac{1}{1-e^{-\theta/2}}\left(1 - e^{-\frac{\theta}{2}\left(1-e^{-\left(G(x)/\bar{G}(x)\right)^{\beta}}\right)^{2}}\right) \qquad (1)$$

The corresponding PDF of (1) is defined by

$$f(x) = \frac{\theta\beta}{1-e^{-\theta/2}}\, g(x)\, \frac{G^{\beta-1}(x)}{\bar{G}^{\beta+1}(x)}\, e^{-\left(\frac{G(x)}{\bar{G}(x)}\right)^{\beta}} \left(1 - e^{-\left(G(x)/\bar{G}(x)\right)^{\beta}}\right) e^{-\frac{\theta}{2}\left(1-e^{-\left(G(x)/\bar{G}(x)\right)^{\beta}}\right)^{2}} \qquad (2)$$

where θ and β are positive scales and shape parameters.

The random variable with PDF (2) is denoted by $X \sim TROW - G(\theta, \beta, \omega)$. By inverting (1), the form of quantile function related to the TROW − G family is

$$Q(q) = \psi(q)\left(\frac{G(x)}{\bar{G}(x)}\right)^{-1} \quad ; \; 0 < q < 1 \qquad (3)$$

with

$$\psi(q) = \left[-\ln\left(1 - \left(\frac{-2}{\theta}\ln\left(1 - \left(1 - e^{-\frac{\theta}{2}}\right)q\right)\right)^{\frac{1}{2}}\right)\right]^{\frac{1}{\beta}} \qquad (4)$$

2. THE TROWIE DISTRIBUTION

The PDF and CDF of the one scale-parameter IE distribution are $g(x;\lambda) = \frac{\lambda}{x^2}e^{-\lambda/x}$ and $G(x;\lambda) = e^{-\lambda/x}$ with $x > 0$ and $\lambda > 0$ (see [16]-[18]). By inserting the CDF of IE in (1), the CDF of a new extended version TROWIE can be obtained as

$$F(x) = \frac{1}{1-e^{-\theta/2}}\left(1 - e^{-\frac{\theta}{2}\left(1-e^{-\left(e^{\lambda/x}-1\right)^{-\beta}}\right)^{2}}\right) \qquad (5)$$

By inserting the PDF and CDF of IE in (2), the corresponding PDF follows, is

$$f(x) = \frac{\theta\beta\lambda}{1-e^{-\theta/2}}\, \frac{e^{-\beta\lambda/x}}{x^2\left(1-e^{-\lambda/x}\right)^{\beta+1}}\, e^{-\left(e^{\lambda/x}-1\right)^{-\beta}} \left(1 - e^{-\left(e^{\lambda/x}-1\right)^{-\beta}}\right) e^{-\frac{\theta}{2}\left(1-e^{-\left(e^{\lambda/x}-1\right)^{-\beta}}\right)^{2}} \qquad (6)$$

The random variable with PDF (6) is denoted by $X \sim TROWIE(\theta, \beta, \lambda)$, and the PDF's natural logarithm of (6) is

$$\ln(f(x)) = \ln\left(\frac{\theta\beta\lambda}{1-e^{-\theta/2}}\right) - \frac{\beta\lambda}{x} - 2\ln(x) - (\beta+1)\ln\left(1-e^{-\lambda/x}\right) - \left(e^{\lambda/x}-1\right)^{-\beta} + \ln\left(1 - e^{-\left(e^{\lambda/x}-1\right)^{-\beta}}\right) - \frac{\theta}{2}\left(1 - e^{-\left(e^{\lambda/x}-1\right)^{-\beta}}\right)^{2} \qquad (7)$$

With the aid of the following extended special formulas,

$$(S1)\;\; e^{-z} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} z^m$$

$$(S2)\;\; (1-z)^a = \sum_{m=0}^{\infty} (-1)^m \binom{a}{m} z^m;\; |z| < 1, a > 0$$

$$(S3)\;\; (1-z)^{-a} = \sum_{m=0}^{\infty} \binom{m+a-1}{m} z^m; |z| < 1, a > 0$$

The essential expanded form of the CDF (5) is

$$F(x)^e = \frac{1}{1-e^{-\theta/2}}\left(1 - \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j+k} j^k}{i!\,k!}\binom{2i}{j}\left(\frac{\theta}{2}\right)^i \left(e^{\lambda/x}-1\right)^{-\beta k}\right) \qquad (8)$$

The corresponding essential expanded form of the PDF (6) is

$$f(x)^e = \sum_{k,\ell=0}^{\infty} \Upsilon^{(k,\ell)}\, \frac{\lambda}{x^2} e^{-\lambda[\beta(k+1)+\ell]/x} \qquad (9)$$

where

$$\Upsilon^{(k,\ell)} = \frac{\theta\beta}{1-e^{-\theta/2}} \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j+k}}{i!\,k!}(j+1)^k \binom{2i+1}{j}\left(\frac{\theta}{2}\right)^i \binom{\beta(k+1)+\ell}{\ell} \qquad (10)$$

Plots of the CDF and PDF of the TROWIE distribution for few parameter values are shown in Figures 1 and 2. Figure 1 clearly demonstrates the common CDF's features. Figure 2 displays some PDF shapes including decreasing, right-skewed, left-skewed, reversed J, and symmetric, which denote the suitability of TROWIE to model different positive data.

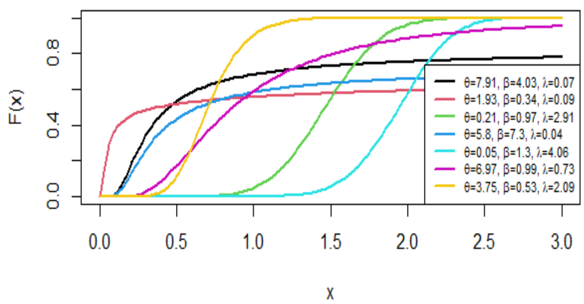

Figure 1. A plot of the CDF with some particular parameter values.

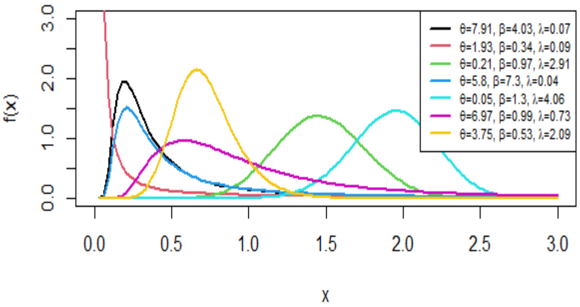

Figure 2. A plot of the PDF with some particular parameter values.

The other essential properties of TROWIE distribution are discussed in the following sub-sections.

2.1. Linear Representation and Related Measures Capital Each Word of Title

The $f(x)^e$ in (9) can rewrite as

$$f(x)^e = \sum_{k,\ell=0}^{\infty} \Upsilon^{(k,\ell)} \frac{\lambda[\beta(k+1)+\ell]}{x^2\,[\beta(k+1)+\ell]} e^{-\lambda[\beta(k+1)+\ell]/x}$$

$$= \sum_{k,\ell=0}^{\infty} \Upsilon^{(k,\ell)} \frac{1}{\beta(k+1)+\ell} \frac{\lambda[\beta(k+1)+\ell]}{x^2} e^{-\lambda[\beta(k+1)+\ell]/x}$$

Therefore

$$f(x)^e = \sum_{k,\ell=0}^{\infty} \Upsilon^{(k,\ell)} \frac{1}{\beta(k+1)+\ell}\, g(x;\lambda[\beta(k+1)+\ell])_{IE} \tag{11}$$

where $g(x;\lambda[\beta(k+1)+\ell])_{IE}$ represents the PDF of the traditional IE with parameter $\lambda[\beta(k+1)+\ell]$. That is the PDF of TROWIE is expressed as a linear combination of the IE distribution.

Regards to (11), r^{th}non-central moment can be found as

$$E(X^r) = \sum_{k,\ell=0}^{\infty} \Upsilon^{(k,\ell)} \frac{1}{\beta(k+1)+\ell} E(X^r)_{g(x;\lambda[\beta(k+1)+\ell])_{IE}}$$

where $E(X^r)_{g(x;\lambda[\beta(k+1)+\ell])_{IE}}$ represents the r^{th}non-central moment of IE with parameter $\lambda[\beta(k+1)+\ell]$. Thus, with $r < 1$

$$E(X^r) = \sum_{k,\ell=0}^{\infty} \Upsilon^{(k,\ell)} \lambda^r (\beta(k+1)+\ell)^{r-1}\Gamma(1-r) \tag{12}$$

Further, based on the linear representation, the characteristic function of the TROWIE is given by

$$\varphi_X(t) = \sum_{k,\ell=0}^{\infty} \Upsilon^{(k,\ell)} \frac{1}{\beta(k+1)+\ell} \varphi_X(t)_{g(x;\lambda[\beta(k+1)+\ell])_{IE}}$$

where $\varphi_X(t)_{g(x;\lambda[\beta(k+1)+\ell])_{IE}}$ is the characteristic function of IE with parameter $\lambda[\beta(k+1)+\ell]$. Thus, the form of the characteristic function is

$$\varphi_X(t) = \sum_{k,\ell=0}^{\infty} \Upsilon^{(k,\ell)} \frac{2\sqrt{-it\lambda[\beta(k+1)+\ell]}}{\beta(k+1)+\ell} K_1\left(2\sqrt{-it\lambda[\beta(k+1)+\ell]}\right) \tag{13}$$

where $K_1\left(2\sqrt{-it\lambda[\beta(k+1)+\ell]}\right)$ is the Bessel's modified function, $K_\alpha(v) = \frac{1}{2}\int_0^\infty y^{\alpha-1} e^{-\frac{v}{2}(y+y^{-1})} dy$, with $\alpha = 1$ and $v = 2\sqrt{-it\lambda[\beta(k+1)+\ell]}$.

2.2. Quantile Function and Related Measures

Recall (3) with baseline distribution IE, $\psi(q)\left(\frac{G(x)}{\bar{G}(x)}\right)^{-1} = \psi(q)\left(e^{\lambda/x}-1\right) \Rightarrow \psi^{-1}(q) = e^{\lambda/x}-1 \Rightarrow e^{\lambda/x} = 1+\psi^{-1}(q)$. After taking the natural logarithm for both sides, the quantile function is

$$Q(q) = \lambda\left[\ln\left(1+\psi^{-1}(q)\right)\right]^{-1} \tag{14}$$

where

$$\psi^{-1}(q) = \left[-\ln\left(1-\left(\frac{-2}{\theta}\ln(1-\left(1-e^{-\theta/2}\right)q)\right)^{1/2}\right)\right]^{-1/\beta} \tag{15}$$

The median and simulated data of TROWIE random variable can be attained respectively via putting $q = 1/2$ and replacing q with U where U~Uniform (0,1).

2.3. Reliability Measures

The reliability measures of any lifetime equipment are the core tools for analyzing aging and associated aspects. The most commonly used measures in real-life data analysis and especially in reliability engineering are [19] reliability $R(x) = 1-F(x)$, hazard $h(x) = f(x)/R(x)$, cumulative hazard $ch(x) = -\ln(R(x))$, and reverse hazard $rh(x) = f(x)/F(x)$ functions that employed to assess how well an item (component or system) performs. Related to CDF and PDF of $TROWIE$ in (5) and (6), the four mentioned measures can easily attend. Figures 3 and 4 display the plots of the reliability and hazard of $TROWIE$ distribution for some parameter values. Figure 3 clearly demonstrates the common features of the reliability function. Figure 4 displays some hazard shapes including increasing, decreasing, right-skewed, J, and reversed J, which indicate the suitability of $TROWIE$ to analyze various sorts of lifetime data.

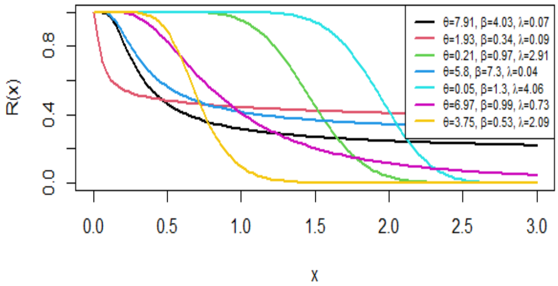

Figure 3. A plot of the reliabilty with some particular parameter values.

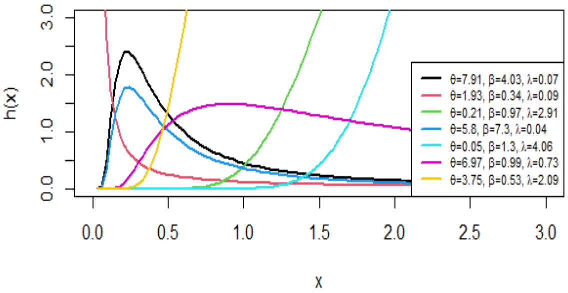

Figure 4. A plot of the hazard with some particular parameter values.

2.4. Entropies

Information theory uses entropy to describe the degree of uncertainty linked with random variable. The entropy of $TROWIE$ random variable can be measured in different ways, two of them are considered in this sub-section.

The first entropy (Shannon entropy) with formula $E_1 = -E\left(\ln\left(f(x)\right)\right)$ [15] and [19] can be achieved from (7) as

$$\begin{aligned} E_1 &= \ln\left(\frac{1-e^{-\theta/2}}{\theta\beta\lambda}\right) + \beta\lambda E\left(\frac{1}{X}\right) + 2E(\ln(X)) \\ &+(\beta+1)E\left(\ln\left(1-e^{-\lambda/X}\right)\right) \\ &-E\left(\ln\left(1-e^{-\left(e^{\lambda/X}-1\right)^{-\beta}}\right)\right) + E\left(\left(e^{\lambda/X}-1\right)^{-\beta}\right) \\ &+\frac{\theta}{2}E\left(\left(1-e^{-\left(e^{\lambda/X}-1\right)^{-\beta}}\right)^2\right) \end{aligned} \quad (16)$$

where $E\left(\frac{1}{X}\right)$ as in (12) with $r = -1$, and $E(\ln(X))$ can be obtained related to PDF (9) as

$$\begin{aligned} E(\ln(X)) &= \int_0^\infty \ln(x)\, f(x)^e\, dx \\ &= \sum_{k,\ell=0}^{\infty} \Upsilon^{(k,\ell)} \int_0^\infty \ln(x)\, \frac{\lambda}{x^2} e^{-\lambda[\beta(k+1)+\ell]/x} dx \end{aligned}$$

After using the transformation $u = \lambda[\beta(k+1)+\ell]/x \longrightarrow x = \lambda[\beta(k+1)+\ell]/u$ and $dx = -\lambda[\beta(k+1)+\ell]\frac{1}{u^2}du$, and recalling the special formula $S4$,

$$(S4) \quad \int_0^\infty z^{s-1}\ln(z)\, e^{-mz} dz = m^{-s}\Gamma(s)(\psi(s)-\ln(m))$$

Thus

$$E(\ln(X)) = \sum_{k,\ell=0}^{\infty} \Upsilon^{(k,\ell)} \frac{1}{\beta(k+1)+\ell} [\ln(\lambda[\beta(k+1)+\ell]) - \psi(1)] \quad (17)$$

where digamma function $\psi(1) = \int_0^\infty \ln(u)\, e^{-u}\, du \cong -0.577$.More over, with the assistance of $S1 - S3$ and extension formula $S5$,

$$(S5) \quad \ln(1-z) = -\sum_{m=1}^{\infty} \frac{1}{m} z^m\ ; |z| < 1$$

The other expectations in (16) can be obtained with some simple mathematical steps as

$$E\left(\ln\left(1-e^{-\lambda/X}\right)\right) = \sum_{m=1}^{\infty}\sum_{s=0}^{\infty} \frac{(-1)^{s+1}}{s!} m^{s-1}\lambda^s E(X^{-s}) \quad (18)$$

$$E\left(\left(e^{\lambda/X}-1\right)^{-\beta}\right) = \sum_{m,s=0}^{\infty} \frac{(-1)^s}{s!}\binom{m+\beta-1}{m} (\lambda(m+\beta))^s E(X^{-s}) \quad (19)$$

$$E\left(\ln\left(1-e^{-\left(e^{\lambda/X}-1\right)^{-\beta}}\right)\right) = \sum_{v=1}^{\infty}\sum_{t,m,s=0}^{\infty} \frac{(-1)^{t+s+1}}{t!\,s!} v^{t-1}\binom{m+\beta t-1}{m}(\lambda(m+\beta t))^s\, E(X^{-s}) \quad (20)$$

$$E\left(\left(1-e^{-\left(e^{\lambda/X}-1\right)^{-\beta}}\right)^2\right) = \sum_{v,t,m,s=0}^{\infty} \frac{(-1)^{v+t+s}}{t!\,s!} v^t \binom{2}{v}\binom{m+\beta t-1}{m}(\lambda(m+\beta t))^s\, E(X^{-s}) \quad (21)$$

where $E(X^{-s})$ as in (12) with $r = -s$.

To obtain the second entropy (relative entropy), consider $f(x)$ and $f_1(x)$ to be the PDFs of two independent random variables following TROWIE respectively with parameters (θ, β, λ) and $(\theta_1, \beta_1, \lambda_1)$, then

$$\begin{aligned} E_2 &= E\left(\ln\left(\frac{f(x)}{f_1(x)}\right)\right) \\ &= E\left(\ln\left(f(x)\right)\right) - E\left(\ln\left(f_1(x)\right)\right) \end{aligned} \quad (22)$$

where $E\left(\ln\left(f(x)\right)\right)$ and $E\left(\ln\left(f_1(x)\right)\right)$ can easily be obtained as mentioned before.

2.5. Reliability Stress Strength (RSS) Model

The RSS is a term used in reliability theory to describe the life of an experimental unit under random stress (Z) and random strength (X). The system operates well if $X > Z$, or when the strength exceeds the stress (see [19] and [20]), which means that the experimental unit breaks instantly when Z applied to it exceeds X. As a result, the measure of experimental unit reliability is $P(X > Z)$. The RSS of two independent variables $X \sim TROWIE(\theta, \beta, \lambda)$ and $Z \sim TROWIE(\theta_1, \beta_1, \lambda_1)$ can be obtained with recalling "(8)" with $(\theta_1, \beta_1, \lambda_1)$ by $RSS = E(F_Z(x)^e)$ as

$$RSS = \frac{1}{1-e^{-\theta_1/2}} \left(1 - \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j+k}}{i!\,k!} j^k \binom{2i}{j} \left(\frac{\theta_1}{2}\right)^i E\left(\left(e^{\lambda_1/X} - 1\right)^{-\beta_1 k} \right) \right)$$

Based on (19) with λ_1 and $\beta_1 k$ respectively instead of λ and β, the RSS is

$$RSS = \frac{1}{1-e^{-\theta_1/2}} \left(1 - \sum_{i,j,k,m,s=0}^{\infty} \frac{(-1)^{i+j+k+s}}{i!\,k!\,s!} j^k \binom{2i}{j} \left(\frac{\theta_1}{2}\right)^i \binom{m+\beta_1 k - 1}{m} (\lambda_1 (m+\beta_1 k))^s E(X^{-s}) \right) \quad (23)$$

2.6. Maximum Likelihood Estimators (MLE)

Regarding PDF (6), for a complete random sample $(x_1, x_2, \ldots, x_n)$ of size n, the natural logarithm likelihood function related to $TROWIE$ with the vector of parameters $\Delta = (\theta, \beta, \lambda)^T$ is

$$\ell(\Delta|\underline{x}) = n \ln\left(\frac{\theta\beta\lambda}{1-e^{-\theta/2}}\right) - \beta\lambda \sum_{i=1}^{n} \frac{1}{x_i} - 2\sum_{i=1}^{n} \ln(x_i) - (\beta+1) \sum_{i=1}^{n} \ln\left(1 - e^{-\frac{\lambda}{x_i}}\right) - \sum_{i=1}^{n} \left(e^{\lambda/x_i} - 1\right)^{-\beta} + \sum_{i=1}^{n} \ln\left(1 - e^{-\left(e^{\lambda/x_i}-1\right)^{-\beta}}\right) - \frac{\theta}{2} \sum_{i=1}^{n} \left(1 - e^{-\left(e^{\lambda/x_i}-1\right)^{-\beta}}\right)^2 \quad (24)$$

The MLE of three parameters can be obtained numerically through solving the not closed forms of nonlinear differential equations $\frac{\partial \ell(\Delta|\underline{x})}{\partial \theta} = 0$, $\frac{\partial \ell(\Delta|\underline{x})}{\partial \beta} = 0$, $\frac{\partial \ell(\Delta|\underline{x})}{\partial \lambda} = 0$.

3. SIMULATION STUDY

A simulation study is conducted to assess the performance of MLE to study how these estimators of the unknown parameters behave for several sample sizes and different parameter combinations.

For each sample size, 3000 random samples of $TROWIE(\theta, \beta, \lambda)$ are generated via simulated formula involved in (14). The MLEs are obtained using the iterative technique available in program R (optim function). The performance of MLE is evaluated with Average Estimates (AE) and Mean Square Error (MSE),

$AE(\hat{\Delta}) = \frac{1}{3000} \sum_{i=1}^{3000} (\hat{\Delta}_i)$; $\Delta = \theta, \beta,$ or λ and

$MSE(\hat{\Delta}) = \frac{1}{3000} \sum_{i=1}^{3000} (\hat{\Delta}_i - \Delta)^2$.

The simulation outcomes related to six different combinations of parameters (for the PDF shapes, see Figure 2 with four sample sizes are shown in Table 1. It is noted that as sample size increases, the values AE tend to be close to the true values, and MSE values seem to be decreasing as expected, demonstrating the consistency of the estimators.

TABLE 1. Values of AE and MSE related to the parameters

Δ	*Tr.*	*AE*	*MSE*	*Tr.*	*AE*	*MSE*
			$n = 15$			
θ	7.91	8.327320	8.087636	1.93	2.012803	3.082864
β	4.03	4.441194	1.374382	0.34	0.374973	0.009585
λ	0.07	0.069945	0.000015	0.09	0.098021	0.001762
θ	0.21	0.472502	7.274642	5.8	5.734592	1.749384
β	0.97	1.054819	0.054117	7.3	8.025454	3.670264
λ	2.91	3.051733	0.589844	0.04	0.039873	8×10^{-7}
θ	6.97	6.515750	4.235454	3.75	3.517010	8.892786
β	0.99	1.131390	0.100377	0.53	0.594638	0.030576
λ	0.73	0.698358	0.020149	2.09	2.139330	0.934870
			$n = 30$			
θ	7.91	8.232776	6.113207	1.93	2.023179	1.867331
β	4.03	4.216355	0.529582	0.34	0.354612	0.003453
λ	0.07	0.070072	0.000009	0.09	0.094913	0.000804
θ	0.21	0.463850	5.850661	5.8	5.803626	1.325745
β	0.97	1.002085	0.020135	7.3	7.657964	1.505122
λ	2.91	3.023564	0.421412	0.04	0.039949	5×10^{-7}
θ	6.97	6.482723	3.456168	3.75	3.700618	5.965475
β	0.99	1.068240	0.039921	0.53	0.555355	0.012235
λ	0.73	0.702371	0.012082	2.09	2.156232	0.569826
			$n = 60$			
θ	7.91	8.122817	3.651627	1.93	1.990555	1.180167
β	4.03	4.114413	0.243656	0.34	0.345894	0.001532
λ	0.07	0.070105	0.000006	0.09	0.093067	0.000408
θ	0.21	0.372998	4.254743	5.8	5.846645	0.977229
β	0.97	0.980548	0.009239	7.3	7.479026	0.693581
λ	2.91	2.987964	0.295312	0.04	0.039993	3×10^{-7}
θ	6.97	6.592483	3.418955	3.75	3.739122	3.585034
β	0.99	1.033369	0.019687	0.53	0.540339	0.005700
λ	0.73	0.711428	0.009384	2.09	2.136193	0.301010
			$n = 120$			
θ	7.91	8.125081	3.007407	1.93	1.962725	0.677580
β	4.03	4.061155	0.114222	0.34	0.342609	0.000712
λ	0.07	0.070149	0.000004	0.09	0.091480	0.000215
θ	0.21	0.309839	2.678956	5.8	5.929748	0.759765
β	0.97	0.972393	0.004317	7.3	7.378719	0.320927
λ	2.91	2.955075	0.176735	0.04	0.040035	2×10^{-7}
θ	6.97	6.719152	3.111888	3.75	3.733695	2.118044
β	0.99	1.013170	0.011206	0.53	0.534231	0.002852
λ	0.73	0.719259	0.007652	2.09	2.116098	0.167366

4. APPLICATIONS TO REAL DATA

In this section, the flexibility of $TROWIE$ distribution is proved by analyzing medical and engineering datasets given as follows.

Medical dataset: The data consist of 120 daily death cases of COVID-19 in Iraq from 1st August to 28th November 2021 accessible at the Iraqi Ministry of Health, Public Health's Directorate [15].

Engineering dataset: The data consists of 38 values of the lifetime period operating the inverter split air conditioner devices before failure happens, taken from an Iraqi Home Electronics Company from 2018 to 2022, are: 1.28, 2.20, 1.26, 27.00, 7.25, 28.17, 12.29, 10.00, 3.05, 3.20, 7.15, 5.27, 6.22, 5.10, 4.12, 8.15, 4.19, 8.03, 6.25, 1.15, 11.00, 5.27, 7.24, 5.12, 8.15, 8.20, 3.13, 1.21, 5.20, 3.18, 6.12, 2.10, 7.24,1.29, 2.22, 3.17, 28.22, 19.10. The integer number denotes the months and the decimal number denotes the days.

Tables 2 and 3 include summary descriptive statistics that can be used to infer the nature of the datasets. The first set of data is right-skewed and platykurtic, and the second is right-skewed and leptokurtic. The $TROWIE$'s fitting behavior is compared with other extended IE related to five families, namely Beta-G, Kumaraswamy-G, Exponentiated Generalized-G, Weibull-G, and Gompertz-G, respectively denoted by $BeIE$, $KuIE$, $EGIE$, $WeIE$, and $GoIE$. It is important to point out that the p-values of the goodness of fit test Kolmogorov-Smirnov (K-S) associated with fitted $TROWIE$ and competitive distributions are all significant values (greater than 0.05). The traditional distribution IE is also included in the comparison. For the comparing process, the negative estimated log-likelihood $(-\hat{\ell})$, and common information criteria (IC) related to Akaike (AIC), Consistent Akaike $(CAIC)$, Bayesian (BIC), Hanan and Quinn $(HQIC)$ (see [15] and [19]) are employed. The distribution with the lowest values of these criteria is the best fit for the considered dataset. Tables 4 - 7 exhibit the outcomes of various MLE values and IC values. The $TROWIE$ distribution clearly has the lowest values of IC making it the best suited to represent COVID-19 medical data and lifetime period engineering data compared to other competitor distributions.

TABLE 2. Descriptive Statistics related to medical data

Stat.	Mean	Median	Min.	Max.	Sk.	Ku.
Val.	42.758	36.5	7.00	87.00	0.39219	-1.11064

TABLE 3. Descriptive Statistics related to engineering data

Stat.	Mean	Median	Min.	Max.	Sk.	Ku.
Val.	7.33	5.27	1.15	28.22	2.07	3.91

TABLE 4. MLE values related to medical data

Dist.	θ	β	λ
TROWIE	0.084405	1.201666	24.000442
BeIE	11.94404	4.397245	11.041476
KuIE	7.794827	6.321404	10.946719
EGIE	4.528481	1.640757	57.672792
WeIE	4.050667	1.640752	10.005726
GoIE	0.005616	2.425328	4.1478026
IE	---	---	34.001800

TABLE 5. MLE values related to engineering data

Dist.	θ	β	λ
TROWIE	0.08389882	0.67202273	2.84322329
BeIE	0.13633080	3.52900290	32.2836614
KuIE	1.08488600	1.74442600	4.71216500
EGIE	4.22806020	0.15457620	28.5583330
WeIE	2.38650400	1.84243600	1.16782200
GoIE	0.85214660	0.66453520	3.74076890
IE	---	---	3.64531100

TABLE 6. Information criteria for fitting medical data

Dist.	$-\hat{\ell}$	AIC	CAIC	BIC	HQIC
TROWIE	519.715	1045.43	1045.64	1053.79	1048.83
BeIE	523.711	1053.42	1053.63	1061.78	1056.82
KuIE	522.228	1050.46	1050.66	1058.82	1053.85
EGIE	523.410	1052.82	1053.03	1061.18	1056.22
WeIE	519.946	1045.89	1046.10	1054.25	1049.29
GoIE	522.576	1051.15	1051.36	1059.51	1054.55
IE	571.603	1145.21	1145.24	1147.99	1146.34

TABLE 7. Information criteria for fitting engineering data

Dist.	$-\hat{\ell}$	AIC	CAIC	BIC	HQIC
TROWIE	109.331	224.662	225.368	229.575	226.410
BeIE	109.619	225.238	225.944	230.150	226.986
KuIE	110.562	227.124	227.830	232.037	228.872
EGIE	109.583	225.166	225.872	230.079	226.914
WeIE	109.459	224.918	225.624	229.831	226.666
GoIE	109.356	224.711	225.417	229.624	226.459
IE	113.050	228.099	228.210	229.737	228.682

5. CONCLUSIONS

A new extended version of inverse exponential distribution named Truncated Rayleigh Odd Weibull Inverse Exponential is proposed. The essential statistical functions, properties, entropies, linear representation, measures of reliability, and stress strength are derived. The shapes of the density and hazard functions are investigated. The density function may be decreasing, right-skewed, left-skewed, reversed J, and symmetric. Further, hazard function may be increasing, decreasing, right-skewed, J, and reversed J, which indicate the suitability of the new extended to analyze various lifetime data. The three unknown parameters are estimated with the method of maximum likelihood and a simulation study examines the accuracy of the estimates. Two applications based on real medical and engineering datasets are considered. Based on the numerical results of different information criteria, new distribution is preferable to several well-known comparable models including the traditional inverse exponential distribution due to its flexible features. This adaptability might make it possible to use the new distribution in more application areas. Other parameter estimation techniques, like least squares, weighted least squares, moments, and Bayesian might also be taken into consideration as a potential field for future research.

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Arabic Abstract

تم اقتراح توزيع احتمالي جديد يسمى Truncated Rayleigh Odd Weibull Inverse Exponential كتوسيع للتوزيع الأسي المعكوس التقليدي. تم اشتقاق الخصائص الإحصائية الأساسية بما في ذلك العزوم، الدالة الكمية، التمثيل الخطي، مقاييس المعولية، الانتروبيا، وأنموذج معولية الإجهاد المتانة. تم تقدير المعلمات الثلاث المجهولة بطريقة الامكان الاعظم وتقديم دراسة محاكاة لفحص دقة التقديرات. تم النظر في تطبيقين يعتمدان على مجموعتين من البيانات الحقيقية - الطبية والهندسية. نظرًا لميزاته المرنة، يُفضل التوزيع الموسع الجديد على عدد من النماذج المماثلة المعروفة.