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A New Subclass of Analytic Functions on Unit Disk Defined by Using Integral Operator

Ali Musaddak Delphi[1*]

[1] *Department of Mathematics, College of Science for Women, University of Baghdad, Iraq*

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ABSTRACT

The objective of this paper is to construct a new subclass of univalent analytic functions using integral operator inside an open unit disk U . The method for proving these theorems, which are utilized to derive new findings on this topic, relies on the Lemma 1.1 and Lemma 1.2 that are stated in this study. The new findings of coefficient bounds for new subclass were used to obtain the theorems of The Growth and distortion , Extreme points and Hadamared product of functions. The novelty of this work adds to the body of knowledge already available on the convolution of univalent analytic function and integral operator.

1. INTRODUCTION

Complex analysis is a rich and multifaceted field with roots in the eighteenth century. It has applications not just to other disciplines of analysis but also to other fields of mathematics and science as a whole. The theory of conformal representation and the geometric function theory of analytic functions are two significant areas of complex analysis. The latter is developed at the turn of the twentieth century and deals with geometrical characteristic of analytic functions; it is still important area of study today. The writings of Bieberbach in 1916 [1] about coefficient problem of univalent analytic functions are among the first noteworthy studies that address subjects from this realm. In k −dimensional complex coordinate space, he was able to drive certain results regarding the range of possible values at the point $d_1, d_2, \cdots d_k$. The best value of n_k is k where $\mathbb{S}(\varpi) = \varpi + d_2\varpi^2 + \cdots d_k\varpi^k + \cdots$, $|d_k| \le n_k$; this equality holds if and only if $\mathbb{S}(\varpi) = \frac{\varpi}{(1-\varpi)^2}$ or one of its rotation. This statement is known as the Biberbach conjecture. In 1923 Lowner [2] proved the Biberbach conjecture for $n = 3$. Finally, in 1985 De Branges [3] proved the Biberbach conjecture for all coefficients with the help of hypergeometric functions. This affirmation elevated the field of geometric function theory to one of the ever growing areas of possible research. Since the Biberbach conjecture was difficult to settle, several authors have considered classes defined by geometric conditions. Notable among them are the classes of starlike functions, convex functions and closed to convex functions. This problem persisted as a difficulty for many years, spurring the creation of intricate and new research techniques that laid the groundwork for the subsequent production of hundreds of articles on the subject. In the area of coefficient bounds for univalent analytic functions on unit disk, there are two research gaps. The first is numerical and computational in nature. As computational tools become more powerful, effective numerical method for computing coefficient bounds must be developed. This can involve using optimization techniques to enhance the bounds, which will help us understand the geometric properties of univalent function better. The second area of research gap is the extension of the study of

*Corresponding Author Institutional Email:
ali.m@csw.uobaghdad.edu.iq (Ali Musaddak Delphi)

coefficient bounds to univalent analytic functions in higher-dimensional spaces, like the unit ball or unit polydisk in $\mathbb{C}^n$. The first research gap is the subject of this paper, which aims to create a new subclass for this topic and apply an integral operator technique to improve the coefficient bounds. Now, let $\mathcal{M}$ be the class of functions given by

$$\mathbb{S}(\varpi) = \varpi + \sum_{k=2}^{\infty} d_k \varpi^k, \qquad (1.1)$$

Which in the open unit disk $U = \{\varpi \in \mathbb{C}: |\varpi| < 1\}$ are analytic and univalent

If $\mathbb{S}$ is given by (1.1) and $\mathbb{T}$ is defined by

$$\mathbb{T}(\varpi) = \varpi + \sum_{k=2}^{\infty} b_k \varpi^k, \qquad (1.2)$$

is in $\mathcal{M}$, then the convolution (Hadamard product) of $\mathbb{S}$ and $\mathbb{T}$ in U is defined by

$$(\mathbb{S} * \mathbb{T})(\varpi) = \varpi + \sum_{k=2}^{\infty} d_k b_k \varpi^k, \qquad (1.3)$$

For $\mathbb{S}$ in $\mathcal{M}$ is starlike of order $\varrho(0 \le \varrho \le 1)$ if $Re\left\{\frac{\varpi \mathbb{S}'}{\mathbb{S}}\right\} > \varrho$ and is convex of order ϱ if $Re\left\{1 + \frac{\varpi \mathbb{S}''}{\mathbb{S}'}\right\} > \varrho$, respectively symbolizes by $\mathbb{S} \in \mathcal{S}^*_{\mathcal{M}}(\varrho)$ and $\mathbb{S} \in \mathcal{K}_{\mathcal{M}}(\varrho)$ for $|\varpi| < 1$.

For $\mathbb{S} \in \mathcal{M}$, the following integral operator that follows was defined by Al-Shaqsi[4]:

$$L_\delta^\ell = (1+\delta)^\ell \Phi_\ell(\delta;\varpi) * \mathbb{S}(\varpi)$$

$$= -\frac{(1+\delta)^\ell}{\Gamma(\ell)} \int_0^1 t^{\delta-1} \log\left(\frac{1}{t}\right)^{\ell-1} \mathbb{S}(\varpi)\, dt, \quad (\delta > 0, \ell > 1, \varpi \in U) \qquad (1.4)$$

Where Γ standarts for the usual gamma function, $\Phi_\ell(\delta;\varpi)$ is the well-known generalization of the Riemann-zeta and polylogarithm functions, or the sth polylogarithm function, given by

$$\Phi_\ell(\delta;\varpi) = \sum_{k=2}^{\infty} \frac{\varpi^k}{(k+\delta)^\ell}$$

Where all terms other than $k + \delta = 0$ is eliminated. Also, the Koebe function is $\Phi_{-1}(0;\varpi) = \sum_{k=1}^{\infty} \frac{\varpi}{(k+\delta)^\ell}$

It can be said that the series expansion of the operator $L_\delta^\ell \mathbb{S}(\varpi)$ given by (1.4) have the following expansions:

$$L_\delta^\ell \mathbb{S}(\varpi) = \varpi + \sum_{k=2}^{\infty} \left(\frac{1+\delta}{k+\delta}\right)^\ell d_k \varpi^k$$

A new subclass $\mathcal{D}(\delta, \ell, \gamma, \tau, \epsilon)$ of $\mathbb{S} \in \mathcal{M}$ is now defined, and it meets the conditions listed below:

$$Re\left\{\frac{\left(L_\delta^\ell \mathbb{S}(\varpi)\right)' + (1+2\gamma)\varpi\left(L_\delta^\ell \mathbb{S}(\varpi)\right)'' + \gamma\varpi^2\left(L_\delta^\ell \mathbb{S}(\varpi)\right)'''}{\left(L_\delta^\ell \mathbb{S}(\varpi)\right)' + \gamma z\left(L_\delta^\ell \mathbb{S}(\varpi)\right)''}\right\} \ge \tau\left|\frac{\left(L_\delta^\ell \mathbb{S}(\varpi)\right)' + (1+2\gamma)\varpi\left(L_\delta^\ell \mathbb{S}(\varpi)\right)'' + \gamma z^2\left(L_\delta^\ell \mathbb{S}(\varpi)\right)'''}{\left(L_\delta^\ell \mathbb{S}(\varpi)\right)' + \gamma z\left(L_\delta^\ell \mathbb{S}(\varpi)\right)''} - 1\right| + \epsilon \qquad (1.5)$$

$$(\delta > 0, \ell > 1, \tau \ge 0, 0 \le \epsilon \le 1, 0 \le \gamma < 1, \varpi \in U)$$

Univalent functions for various subclasses and subjects were examined by several authors, such [5], [6], [7], [8], [9], [10][11], and [12].

The current study aims to develop new results regarding the characteristic of the geometric function of $\mathbb{S} \in \mathcal{M}$ in $\mathcal{D}(\delta, \ell, \gamma, \tau, \epsilon)$ by applying the two lemma mentioned below:

Lemma(1.1)[13]: Let $\mathcal{Y} = \mathfrak{p} + i\mathfrak{q}$ and σ is real number then $Re(\mathcal{Y}) \ge \sigma$ if and only if $|\mathcal{Y} - (1+\sigma)| \le |\mathcal{Y} + (1-\sigma)|$.

Lemma(1.2)[13]: Let $\mathcal{Y} = \mathfrak{p} + i\mathfrak{q}$ and σ, γ are real numbers then $Re(\mathcal{Y}) \ge \sigma|\mathcal{Y} - 1| + \gamma$ if and only if $Re\{\mathcal{Y}(1 + \sigma e^{i\phi}) - \sigma e^{i\phi}\} > \gamma$.

2. COEFFICIENT ESTIMATES

The next theorem provides us with a necessary and sufficient condition for $\mathbb{S} \in \mathcal{D}(\delta, \ell, \gamma, \tau, \epsilon)$

Theorem(2.1): Let $\mathbb{S}$ be defined by (1.1), then $\mathbb{S} \in \mathcal{D}(\delta, \ell, \gamma, \beta, \epsilon)$ if and only if

$$\sum_{k=2}^{\infty} k\big((k-\epsilon) + \tau(k-1)\big)[1 + \gamma(k-1)]\left(\frac{1+\delta}{k+\delta}\right)^\ell d_k \le (1-\epsilon) \qquad (2.1)$$

where

$\delta > 0, \ell > 1, \tau \geq 0, 0 \leq \epsilon < 1, 0 \leq \gamma < 1, \varpi \in U$

Proof: by (1.5), we get

$$Re\left\{\frac{\left(L_\delta^\ell \mathbb{S}(\varpi)\right)' + (1+2\gamma)\varpi\left(L_\delta^\ell \mathbb{S}(\varpi)\right)'' + \gamma\varpi^2\left(L_\delta^\ell \mathbb{S}(\varpi)\right)'''}{\left(L_\delta^\ell \mathbb{S}(\varpi)\right)' + \gamma\varpi\left(L_\delta^\ell \mathbb{S}(\varpi)\right)''}\right\} \geq \tau\left|\frac{\left(L_\delta^\ell \mathbb{S}(\varpi)\right)' + (1+2\gamma)\varpi\left(L_\delta^\ell \mathbb{S}(\varpi)\right)'' + \gamma\varpi^2\left(L_\delta^\ell \mathbb{S}(\varpi)\right)'''}{\left(L_\delta^\ell \mathbb{S}(\varpi)\right)' + \gamma\varpi\left(L_\delta^\ell \mathbb{S}(\varpi)\right)''} - 1\right| + \epsilon$$

Lemma(1.2) then give us

$$Re\left\{\frac{\left(L_\delta^\ell \mathbb{S}(\varpi)\right)' + (1+2\gamma)\varpi\left(L_\delta^\ell \mathbb{S}(\varpi)\right)'' + \gamma\varpi^2\left(L_\delta^\ell \mathbb{S}(\varpi)\right)'''}{\left(L_\delta^\ell \mathbb{S}(\varpi)\right)' + \gamma\varpi\left(L_\delta^\ell \mathbb{S}(\varpi)\right)''}(1 + \tau e^{i\vartheta}) - \tau e^{i\vartheta}\right\} \geq \epsilon$$

$-\pi < \vartheta \leq \pi$, or in the same way,

$$Re\left\{\frac{\left(\left(L_\delta^\ell \mathbb{S}(\varpi)\right)' + (1+2\gamma)\varpi\left(L_\delta^\ell \mathbb{S}(\varpi)\right)'' + \gamma\varpi^2\left(L_\delta^\ell \mathbb{S}(\varpi)\right)'''\right)(1+\tau e^{i\vartheta})}{\left(L_\delta^\ell \mathbb{S}(\varpi)\right)' + \gamma\varpi\left(L_\delta^\ell \mathbb{S}(\varpi)\right)''} - \frac{\tau e^{i\vartheta}\left(\left(L_\delta^\ell \mathbb{S}(\varpi)\right)' + \gamma\varpi\left(L_\delta^\ell \mathbb{S}(\varpi)\right)''\right)}{\left(L_\delta^\ell \mathbb{S}(\varpi)\right)' + \gamma\varpi\left(L_\delta^\ell \mathbb{S}(\varpi)\right)''}\right\} \geq \epsilon \qquad (2.2)$$

Let

$$F(\varpi) = \left(\left(L_\delta^\ell \mathbb{S}(\varpi)\right)' + (1+2\gamma)\varpi\left(L_\delta^\ell \mathbb{S}(\varpi)\right)'' + \gamma\varpi^2\left(L_\delta^\ell \mathbb{S}(\varpi)\right)'''\right)(1+\tau e^{i\vartheta}) - \tau e^{i\vartheta}\left(\left(L_\delta^\ell \mathbb{S}(\varpi)\right)' + \gamma\varpi\left(L_\delta^\ell \mathbb{S}(\varpi)\right)''\right)$$

and $E(\varpi) = \left(L_\delta^\ell \mathbb{S}(\varpi)\right)' + \gamma\varpi\left(L_\delta^\ell \mathbb{S}(\varpi)\right)''$

Lemma(1.1) states that (2.2) is equal to

$|F(\varpi) + (1-\epsilon)E(\varpi)| \geq |F(\varpi) - (1+\epsilon)E(\varpi)|$ for $0 \leq \epsilon < 1$

But

$$\left|\left[1 + \sum_{k=2}^{\infty} k\{1 + (1+2\gamma)(k-1) + \gamma(k-1)(k-2)\}\left(\frac{1+\delta}{k+\delta}\right)^\ell d_k\varpi^{k-1}\right](1+\tau e^{i\vartheta}) - \tau e^{i\vartheta}\left[1 + \sum_{k=2}^{\infty} k\{1+\gamma(k-1)\}\left(\frac{1+\delta}{k+\delta}\right)^\ell d_k\varpi^{k-1}\right] + (1-\epsilon)\left[1 + \sum_{k=2}^{\infty}\{k + \gamma k(k-1)\}\left(\frac{1+\delta}{k+\delta}\right)^\ell d_k\varpi^{k-1}\right]\right|$$

$$= \left|\left[(2-\epsilon) + \sum_{k=2}^{\infty} k(1-\epsilon+k)[1+\gamma(k-1)]\left(\frac{1+\delta}{k+\delta}\right)^\ell d_k\varpi^{k-1}\right] + \tau e^{i\vartheta}\left[\sum_{k=2}^{\infty} k(k-1)[1+\gamma(k-1)]\left(\frac{1+\delta}{k+\delta}\right)^\ell d_k\varpi^{k-1}\right]\right|$$

$$\geq (2-\epsilon) - \sum_{k=2}^{\infty} k(1-\epsilon+k)[1+\gamma(k-1)]\left(\frac{1+\delta}{k+\delta}\right)^\ell d_k|\varpi|^{k-1} - \tau\sum_{k=2}^{\infty} k(k-1)[1+\gamma(k-1)]\left(\frac{1+\delta}{k+\delta}\right)^\ell d_k|\varpi|^{k-1}$$

Also

$|F(\varpi) - (1+\epsilon)E(\varpi)| =$

$$\left|\left[1 + \sum_{k=2}^{\infty} k\{1 + (1+2\gamma)(k-1) + \gamma(k-1)(k-2)\}\left(\frac{1+\delta}{k+\delta}\right)^\ell d_k\varpi^{k-1}\right](1+\tau e^{i\vartheta}) - \tau e^{i\vartheta}\left[1 + \sum_{k=2}^{\infty} k\{1+\gamma(k-1)\}\left(\frac{1+\delta}{k+\delta}\right)^\ell d_k\varpi^{k-1}\right] - (1+\epsilon)\left[1 + \sum_{k=2}^{\infty}\{k + \gamma k(k-1)\}\left(\frac{1+\delta}{k+\delta}\right)^\ell d_k\varpi^{k-1}\right]\right|$$

$$= \left|\left[(-\epsilon) + \sum_{k=2}^{\infty} k\big(k-(1+\epsilon)\big)[1+\gamma(k-1)]\left(\frac{1+\delta}{k+\delta}\right)^\ell d_k\varpi^{k-1}\right] + \tau e^{i\vartheta}\left[\sum_{k=2}^{\infty} k(k-1)[1+\gamma(k-1)]\left(\frac{1+\delta}{k+\delta}\right)^\ell d_k\varpi^{k-1}\right]\right|$$

$$\leq \epsilon + \sum_{k=2}^{\infty} k\big(k-(1+\epsilon)\big)[1+\gamma(k-1)]\left(\frac{1+\delta}{k+\delta}\right)^\ell d_k|\varpi|^{k-1} + \tau\sum_{k=2}^{\infty} k(k-1)[1+\gamma(k-1)]\left(\frac{1+\delta}{k+\delta}\right)^\ell d_k|\varpi|^{k-1}$$

Consequently

$$|F(\varpi) + (1-\epsilon)E(\varpi)| - |F(\varpi) - (1+\epsilon)E(\varpi)| \geq 2(1-\epsilon) - \sum_{k=2}^{\infty} 2k(k-\epsilon)[1+\gamma(k-1)]\left(\frac{1+\delta}{k+\delta}\right)^\ell d_k|\varpi|^{k-1} - \tau\sum_{k=2}^{\infty} 2k(k-1)[1+\gamma(k-1)]\left(\frac{1+\delta}{k+\delta}\right)^\ell d_k|\varpi|^{k-1} \geq 0$$

That is the same as

$$\sum_{k=2}^{\infty} k\big((k-\epsilon) + \beta(k-1)\big)[1+\gamma(k-1)]\left(\frac{1+\delta}{k+\delta}\right)^\ell d_k \leq (1-\epsilon)$$

Conversely, if (2.1) is true, then we need to demonstrate

$$Re\left\{\frac{(1+\beta e^{i\vartheta})\left(\left(L_{\delta}^{\ell}\mathbb{S}(\varpi)\right)'+(1+2\gamma)\varpi\left(L_{\delta}^{\ell}\mathbb{S}(\varpi)\right)''+\gamma\varpi^2\left(L_{\delta}^{\ell}\mathbb{S}(\varpi)\right)'''\right)-\tau e^{i\vartheta}\left(\left(L_{\delta}^{\ell}\mathbb{S}(\varpi)\right)'+\gamma\varpi\left(L_{\delta}^{\ell}\mathbb{S}(\varpi)\right)''\right)}{\left(L_{\delta}^{\ell}\mathbb{S}(\varpi)\right)'+\gamma\varpi\left(L_{\delta}^{\ell}\mathbb{S}(\varpi)\right)''}\right\}\geq\epsilon$$

When selecting values for ϖ on the positive real axis, where $0\leq\varpi=r<1$, the inequality shown above decreases to

$$Re\left\{\frac{(1-\epsilon e^{i\vartheta})+\sum_{k=2}^{\infty}k\left((k-\epsilon e^{i\vartheta})+\beta(k-1)e^{i\vartheta}\right)[1+\gamma(k-1)]\left(\frac{1+\delta}{k+\delta}\right)^{\ell}dr^{k-1}}{1+\sum_{k=2}^{\infty}(k+\gamma k(k-1))\left(\frac{1+\delta}{k+\delta}\right)^{\ell}d_kr^{k-1}}\right\}\geq 0$$

Since $Re\left(-e^{i\vartheta}\right)\geq-\left|e^{i\vartheta}\right|=-1$, the inequality above becomes

$$Re\left\{\frac{(1-\epsilon)+\sum_{k=2}^{\infty}k\left((k-\epsilon)+\tau(k-1)\right)[1+\gamma(k-1)]\left(\frac{1+\delta}{k+\delta}\right)^{\ell}d_kr^{k-1}}{1+\sum_{k=2}^{\infty}(k+\gamma k(k-1))\left(\frac{1+\delta}{k+\delta}\right)^{\ell}d_kr^{k-1}}\right\}\geq 0$$

Letting $r\to 1^-$, the intended conclusion is reached.

Corollary (2.2): Let $\mathbb{S}(\varpi)\in\mathcal{D}(\delta,\ell,\gamma,\tau,\epsilon)$, then

$$\sum_{k=2}^{\infty}d_k\leq\frac{(1-\epsilon)}{2\left((2-\epsilon)+\tau\right)[1+\gamma]\left(\frac{1+\delta}{2+\delta}\right)^{\ell}}\qquad(2.3)$$

and

$$\sum_{k=2}^{\infty}kd_k\leq\frac{(1-\epsilon)}{\left((2-\epsilon)+\tau\right)[1+\gamma]\left(\frac{1+\delta}{2+\delta}\right)^{\ell}}\qquad(2.4)$$

Next, as seen below, the sharpness is satisfied from(2.3)

$$T(\varpi)=\varpi+\sum_{k=2}^{\infty}\frac{(1-\epsilon)}{k\left((k-\epsilon)+\tau(k-1)\right)[1+\gamma(k-1)]\left(\frac{1+\delta}{k+\delta}\right)^{\ell}}\varpi^k$$

3. GROWTH AND DISTORTION THEOREM

Here is the growth and distortion theorems for $\mathbb{S}\in\mathcal{D}(\delta,\ell,\gamma,\tau,\epsilon)$ that can be obtained

Theorem(3.1): Let $\mathbb{S}(\varpi)$ defined by (1.1) be in the subclass $\mathcal{D}(\delta,\ell,\gamma,\tau,\epsilon)$, then

$$r-\frac{(1-\epsilon)}{2\left((2-\epsilon)+\tau\right)[1+\gamma]\left(\frac{1+\delta}{2+\delta}\right)^{\ell}}r^2\leq|\mathbb{S}(\varpi)|\leq r+\frac{(1-\epsilon)}{2\left((2-\epsilon)+\tau\right)[1+\gamma]\left(\frac{1+\delta}{2+\delta}\right)^{\ell}}r^2,$$

$|\varpi|=r<1.$

For the function $\mathbb{S}(\varpi)$ given by

$$\mathbb{S}(\varpi)=\varpi+\frac{(1-\epsilon)}{2\left((2-\epsilon)+\tau\right)[1+\gamma]\left(\frac{1+\delta}{2+\delta}\right)^{\ell}}\varpi^2,$$

the result is sharp.

Proof: Let $\mathbb{S}(\varpi)\in\mathcal{D}(\delta,\ell,\gamma,\tau,\epsilon)$ then by(2.3) in **Corollary (2.2),** we have

$$\sum_{k=2}^{\infty}d_k\leq\frac{(1-\epsilon)}{2\left((2-\epsilon)+\tau\right)[1+\gamma]\left(\frac{1+\delta}{2+\delta}\right)^{\ell}}$$

Hence

$$|\mathbb{S}(\varpi)|\leq|\varpi|+\sum_{k=2}^{\infty}d_k|\varpi|^k=r+r^2\sum_{k=2}^{\infty}d_k\leq r+\frac{(1-\epsilon)}{2\left((2-\epsilon)+\tau\right)[1+\gamma]\left(\frac{1+\delta}{2+\delta}\right)^{\ell}}r^2\qquad(3.2)$$

Similarly

$$|\mathbb{S}(\varpi)|\geq|\varpi|-\sum_{k=2}^{\infty}d_k|\varpi|^k=r-r^2\sum_{k=2}^{\infty}d_k\geq r-\frac{(1-\epsilon)}{2\left((2-\epsilon)+\tau\right)[1+\gamma]\left(\frac{1+\delta}{2+\delta}\right)^{\ell}}r^2\qquad(3.3)$$

from bound (3.2) and (3.3), we get (3.1). ∎

Theorem(3.2): Let the function $\mathbb{S}(\varpi)$ defined by (1.1) be in the subclass $\mathcal{D}(\delta,\ell,\gamma,\tau,\epsilon)$, then

$$1-\frac{(1-\epsilon)}{\left((2-\epsilon)+\tau\right)[1+\gamma]\left(\frac{1+\delta}{2+\delta}\right)^{\ell}}r\leq|\mathbb{S}'(\varpi)|\leq 1+\frac{(1-\epsilon)}{\left((2-\epsilon)+\tau\right)[1+\gamma]\left(\frac{1+\delta}{2+\delta}\right)^{\ell}}r\qquad(3.4)$$

$|\varpi|=r<1.$

For the function $\mathbb{S}(\varpi)$ given by

$$\mathbb{S}(\varpi)=\varpi+\frac{(1-\epsilon)}{\left((2-\epsilon)+\tau\right)[1+\gamma]\left(\frac{1+\delta}{2+\delta}\right)^{\ell}}\varpi^2,$$

the result is sharp

Proof: Let $\mathbb{S}(\varpi) \in \mathcal{D}(\delta, \ell, \gamma, \tau, \epsilon)$ then by(2.4) in **Corollary (2.2),** we have

$$\sum_{k=2}^{\infty} k d_k \leq \frac{(1-\epsilon)}{\big((2-\epsilon)+\tau\big)[1+\gamma]\left(\frac{1+\delta}{2+\delta}\right)^{\ell}}$$

Hence

$$|\mathbb{S}'(\varpi)| \leq |1| + \sum_{k=2}^{\infty} k d_k |\varpi^k|^{k-1} = 1 + r\sum_{k=2}^{\infty} k d_k \leq 1 + \frac{(1-\epsilon)}{\big((2-\epsilon)+\tau\big)[1+\gamma]\left(\frac{1+\delta}{2+\delta}\right)^{\ell}} r \qquad (3.5)$$

Similarly

$$|\mathbb{S}'(\varpi)| \geq |1| - \sum_{k=2}^{\infty} k d_k |\varpi|^{k-1} = 1 - r\sum_{k=2}^{\infty} k d_k \geq 1 - \frac{(1-\epsilon)}{\big((2-\epsilon)+\tau\big)[1+\gamma]\left(\frac{1+\delta}{2+\delta}\right)^{\ell}} r \qquad (3.6)$$

from bound (3.5) and (3.6), we get (3.1). ■

4. EXTREME POINTS

The extreme points theorem for $\mathbb{S} \in \mathcal{D}(\delta, \ell, \gamma, \tau, \epsilon)$ is obtained in the following theorem

Theorem(4.1): Let $\mathbb{S}_1(\varpi) = \varpi$ and

$$\mathbb{S}_k(\varpi) = \varpi + \frac{(1-\epsilon)}{k\big((k-\epsilon)+\tau(k-1)\big)[1+\gamma(k-1)]\left(\frac{1+\delta}{k+\delta}\right)^{\ell}} \varpi^k, for\ k = 2,3, \ldots$$

then $\mathbb{S} \in \mathcal{D}(\delta, \ell, \gamma, \tau, \epsilon)$ if and only if it is able to be stated as

$$\mathbb{S}(\varpi) = \sum_{k=1}^{\infty} \mathcal{T}_k \mathbb{S}_k(\varpi)$$

where $\left(\mathcal{T}_k \geq 0\ and\ \sum_{k=1}^{\infty} \mathcal{T}_k = 1\ \ or\ 1 = \mathcal{T}_1 + \sum_{k=2}^{\infty} \mathcal{T}_k\right)$

proof: Let

$$\mathbb{S}(\varpi) = \sum_{k=1}^{\infty} \mathcal{T}_k \mathbb{S}_k(\varpi) = \varpi + \sum_{k=2}^{\infty} \frac{(1-\epsilon)}{k\big((k-\epsilon)+\tau(k-1)\big)[1+\gamma(k-1)]\left(\frac{1+\delta}{k+\delta}\right)^{\ell}} \mathcal{T}_k \varpi^k$$

then, from **Theorem(2.1)** , we obtain

$$\sum_{k=2}^{\infty} k\big((k-\epsilon)+\tau(k-1)\big)[1+\gamma(k-1)]\left(\frac{1+\delta}{k+\delta}\right)^{\ell} \times \frac{(1-\epsilon)}{k\big((k-\epsilon)+\tau(k-1)\big)[1+\gamma(k-1)]\left(\frac{1+\delta}{k+\delta}\right)^{\ell}} \mathcal{T}_k = (1-\epsilon)\sum_{k=2}^{\infty} \mathcal{T}_k = (1-\epsilon)(1-\mathcal{T}_1) \leq (1-\alpha)$$

It follows that $\mathbb{S} \in \mathcal{D}(\delta, \ell, \gamma, \tau, \epsilon)$ in light of **Theorem(2.1)** Conversely, let $\mathbb{S} \in \mathcal{D}(\delta, \ell, \gamma, \tau, \epsilon)$ then

$$d_k \leq \frac{(1-\epsilon)}{k\big((k-\epsilon)+\tau(k-1)\big)[1+\gamma(k-1)]\left(\frac{1+\delta}{k+\delta}\right)^{\ell}}$$

by setting

$$\mathcal{T}_k = \frac{k\big((k-\epsilon)+\tau(k-1)\big)[1+\gamma(k-1)]\left(\frac{1+\delta}{k+\delta}\right)^{\ell}}{(1-\epsilon)} d_k, for\ k = 2,3, \ldots$$

and

$$\mathcal{T}_1 = 1 - \sum_{k=2}^{\infty} \mathcal{T}_k$$

then

$$\mathbb{S}(\varpi) = \varpi + \sum_{k=2}^{\infty} d_k \varpi^k = \varpi + \sum_{k=2}^{\infty} \frac{(1-\epsilon)}{k\big((k-\epsilon)+\tau(k-1)\big)[1+\gamma(k-1)]\left(\frac{1+\delta}{k+\delta}\right)^{\ell}} \mathcal{T}_k \varpi^k$$

$$= \mathcal{T}_1 \varpi + \sum_{k=2}^{\infty} \mathcal{T}_k \mathbb{S}_k(\varpi) = \sum_{k=1}^{\infty} \mathcal{T}_k \mathbb{S}_k(\varpi),$$

this complete the proof.■

5. HADAMARD PRODUCT

Let $\mathbb{S}_j \in \mathcal{M}$ given by

$$\mathbb{S}_j(\varpi) = \varpi + \sum_{k=2}^{\infty} d_{k,j}\, \varpi^k,$$

$$j = 1,2,\ \ \varpi \in U \qquad (5.1)$$

Then the Hadamard product of $\mathbb{S}_j \in \mathcal{M}$ for $j = 1,2$ is defined by

$$(\mathbb{S}_1 * \mathbb{S}_2)(\varpi) = \varpi + \sum_{k=2}^{\infty} \left(\prod_{j=1}^{2} d_{k,j}\right) \varpi^k,$$

The following theorem includes one of our main findings.

Theorem(5.1): If $\mathbb{S}_j(\varpi) \in \mathcal{D}(\delta, \ell, \gamma, \tau, \epsilon_j)$ for $j = 1,2$, then $(\mathbb{S}_1 * \mathbb{S}_2)(\varpi) \in \mathcal{D}(\delta, \ell, \gamma, \tau, \psi)$

as well as

$$\psi \leq 1 - \frac{\prod_{j=1}^{p=2}(1-\epsilon_j)(\tau+2)}{[1+\gamma]\left(\frac{1+\delta}{2+\delta}\right)^{\ell}\left(\left(\prod_{j=1}^{p=2}(2-\epsilon_j)+\tau\right)\right) - \prod_{j=1}^{p=2}(1-\epsilon_j)}$$

Proof: In order to prove $(\mathbb{S}_1 * \mathbb{S}_2)(\varpi) \in \mathcal{D}(\delta, \ell, \gamma, \tau, \psi)$ it is enough to show

$$\sum_{k=2}^{\infty} \frac{\left(k\big((k-\epsilon_j)+\tau(k-1)\big)[1+\gamma(k-1)]\left(\frac{1+\delta}{k+\delta}\right)^{\ell}\right)}{(1-\epsilon_j)} d_{k,j} \leq 1 \qquad (5.2)$$

For $j = 1,2,$ the Cauchy-Schwarz inequality is used to get

$$\sum_{k=2}^{\infty}[1+\gamma(k-1)]\left(\frac{1+\delta}{k+\delta}\right)^{\ell}\sqrt{\prod_{j=1}^{p=2}\left(\frac{\left(k\left(\begin{matrix}(k-\epsilon_j)+\\ \tau(k-1)\end{matrix}\right)\right)}{(1-\alpha_j)}d_{k,j}\right)} \leq 1 \quad (5.3)$$

To prove $p = 2$, our task is to determine the largest ψ so that

$$\sum_{k=2}^{\infty}\frac{\left(k((k-\psi)+\tau(k-1))[1+\gamma(k-1)]\left(\frac{1+\delta}{k+\delta}\right)^{\ell}\right)}{(1-\psi)}d_{k,j} \leq 1 \quad (5.4)$$

Alternatively, it is equivalent to

$$\sqrt{d_{k,j}} \leq \frac{(1-\psi)}{\left(k((k-\psi)+\tau(k-1))\right)}\frac{\sqrt{\left(k\left(\prod_{j=1}^{p=2}(k-\epsilon_j)+\tau(k-1)\right)\right)}}{\sqrt{\prod_{j=1}^{p=2}(1-\epsilon_j)}} \quad (5.5)$$

Moreover, that is

$$\sqrt{d_{k,1}d_{k,2}} \leq \frac{(1-\psi)}{\left(k((k-\psi)+\tau(k-1))\right)}\sqrt{\prod_{j=1}^{p=2}\frac{\left(k((k-\epsilon_j)+\tau(k-1))\right)}{(1-\epsilon_j)}} \quad (5.6)$$

Finding the largest ψ requires us to go beyond (5.3) in the following manner

$$\frac{\left(k((k-\psi)+\tau(k-1))\right)}{(1-\psi)} \leq [1+\gamma(k-1)]\left(\frac{1+\delta}{k+\delta}\right)^{\ell}\prod_{j=1}^{p=2}\left(\frac{\left(k((k-\epsilon_j)+\tau(k-1))\right)}{(1-\epsilon_j)}\right) \quad (5.7)$$

which is the same as

$$\psi \leq 1 - \frac{\prod_{j=1}^{p=2}(1-\epsilon_j)(\tau(k-1)+k)}{[1+\gamma(k-1)]\left(\frac{1+\delta}{k+\delta}\right)^{\ell}\left(\left(\prod_{j=1}^{p=2}(k-\epsilon_j)+\tau(k-1)\right)\right)-\prod_{j=1}^{p=2}(1-\epsilon_j)}$$

$$\psi = 1 - \frac{\prod_{j=1}^{p=2}(1-\epsilon_j)(\tau(k-1)+k)}{[1+\gamma(k-1)]\left(\frac{1+\delta}{k+\delta}\right)^{\ell}\left(\left(\prod_{j=1}^{p=2}(k-\epsilon_j)+\tau(k-1)\right)\right)-\prod_{j=1}^{p=2}(1-\epsilon_j)} \quad (5.8)$$

Let us now assume that

$$\Phi(k) = 1 - \frac{\prod_{j=1}^{p=2}(1-\epsilon_j)(\tau(k-1)+k)}{[1+\gamma(k-1)]\left(\frac{1+\delta}{k+\delta}\right)^{\ell}\left(\left(\prod_{j=1}^{p=2}(k-\epsilon_j)+\tau(k-1)\right)\right)-\prod_{j=1}^{p=2}(1-\epsilon_j)} \quad (5.9)$$

Since $\Phi'(k) \geq 0$ for $(k \geq 2)$ these produces

$$\psi \leq \Phi(2) = 1 - \frac{\prod_{j=1}^{p=2}(1-\epsilon_j)(\tau+2)}{[1+\gamma]\left(\frac{1+\delta}{k+\delta}\right)^{\ell}\left(\left(\prod_{j=1}^{p=2}(2-\epsilon_j)+\tau\right)\right)-\prod_{j=1}^{p=2}(1-\epsilon_j)} \quad (5.10)$$

thus, this theorem's proof is finished.■

6. CONCLUSION

. In the present paper, a new subclass of univalent analytic function convolution with integral operator on an open unit disk is calculated the coefficient bounds, Growth and distortion theorems. Extreme points and Hadamared product are also obtained for this new subclass. The application of differential equations of order m may benefit from the generalization of this new subclass to a new subclass of multivalent analytic functions that involve a higher derivatives operator in the future.

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Arabic Abstract

الهدف في هذا البحث هو تقديم اصناف جزئية جديدة من الدوال التحليلية احادية التكافؤ المعرفة على قرص الوحدة المفتوح، وطريقة اثبات المبرهنات لإيجاد نتائج جديدة في هذا الموضوع معتمدة على Lemma 1.1 و Lemma 1.2 التي ذكرت في هذه الدراسة. النتائج الجديدة لقيود المعاملات لهذا الصنف الجزئي الجديد استخدمت لإيجاد مبرهنات النمو والتشوه، النقاط المتطرفة وضرب الالتواء للدوال. تضيف حداثة هذا العمل الى المجموعة المتاحة بالفعل حول الالتواء بين الدوال احادية التكافؤ والمؤثر التكاملي.