2026,Volume 3 , Issue 9 , PSIJK (26-33)

PURE SCIENCES International Journal of Kerbala

Pure Sciences International Journal of Kerbala

A Flexible Extended Exponential Distribution With Its Statistical Features, Inference, and Real Application

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P A P E R I N F O

Received: 07.07.2025
Accepted: 21.07.2025
Published: 31.03.2026

Keywords:
traditional exponential, unit half logistic geometric, generator family, statistical features, simulation

A b s t r a c t

This paper introduces a new version of the exponential distribution, offering a more flexible model for real-life data, namely unit half logistic geometric exponential (UHLGeE). The essential statistical functions and features of the two-parameter proposed distribution are discussed. Different techniques, maximum likelihood, ordinary least squares, weighted least squares, and Cramer-Von-Mises minimum distance, are employed to estimate the two unknown parameters. Consequently, extensive simulation experiments are conducted to evaluate the performance of all estimation methods. Lastly, the efficacy and adaptability of the new distribution are illustrated through an analysis of a real data set. Based on the outcomes of the empirical and real applications, it is recommended to adopt and employ UHLGeE in additional applications due to its features and adaptability.

DOI: 10.53851/psijk.v3.i9. 26-33

1. INTRODUCTION

Lifetime distributions are an effective tool for determining the lifespan of devices, systems, and time-to-event data in general. These distributions are commonly employed in various fields, including biology, insurance, engineering, and reliability. With a large range of continuous univariate models, the literature on lifespan distributions is extensive and rapidly expanding. Many modifications to widely used distributions have been introduced over time to handle the complicated nature of real-life data. This includes incorporating one or more extra parameters into the traditional (existing/baseline) distribution. These parameters were valuable for evaluating tail features and enhancing the goodness-of-fit of the resulting distribution. The traditional one-scale exponential (E) distribution is among the most common distributions in various applications. It has a constant hazard function. But, different systems in real life rarely have a stable hazard/risk rate across time (see Singh et al., 2013). Consequently, it appears plausible to assume the hazard as a function of time, which resulted in the creation of an alternative modified/extended model for lifetime data analysis. Numerous distributions have been introduced as extensions of the E distribution. Notable examples of these extensions can be found in the literature, for example, generalized E (Marshall & Olkin, 1997 and Gupta & Kundu, 1999), exponentiated E (Gupta & Kundu, 2001), beta E (Nadarajah & Kotz, 2006), Kumaraswamy E as a special case form the Kumaraswamy Weibull (Cordeiro et al., 2010), extension E (Nadarajah & Haghighi, 2011), gamma-exponentiated E (Ristić & Balakrishnan, 2012), Kumaraswamy generalized exponentiated E (Mohammed, 2014), transmuted generalised E (Khan et al., 2017), exponentiated Weibull-E (Elgarhy et al., 2017), Marshall-Olkin length-biased E (ul Haq et al., 2019), new extension of extended E (Alghamedi et al., 2020), modified extended E (Mahmoud et al., 2022), truncated inverse Rayleigh odd Weibull E (Al-Noor & Sultan, 2024), and unit extended E (Ragab et al., 2024), among others. This paper aims to introduce a new flexible extended version of the E distribution by adding an extra parameter to its traditional version. The rest of the paper is organized as follows: Section 2 contains the statistical methodology for constructing the new distribution. Section 3 discusses the distribution's essential statistical

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features. Section 4 presents four estimation procedures to estimate the unknown parameters. Section 5 provides the empirical and real-life applications. Section 6 provides some concluding remarks.

2. CONSTRUCTING THE NEW DISTRIBUTION

This section explores the new distribution, providing its essential statistical functions: the cumulative distribution function (CDF), probability density function (PDF), reliability function (RF), and hazard function (HF). In this context, the focus is on adding a shape parameter to the traditional version of the E distribution by employing the unit half logistic geometric generator (UHLGe-G) family recently introduced by Najm & Al-Noor (2025), which is characterized by the following CDF and PDF,

$$F(x;\theta,\delta)=\frac{2G(x;\delta)}{\theta+(2-\theta)G(x;\delta)}\ ;\ \theta>0 \qquad (1)$$

$$f(x;\theta,\delta)=\frac{2\theta g(x;\delta)}{(\theta+(2-\theta)G(x;\delta))^2}\ ;\ \theta>0 \qquad (2)$$

where θ is the shape parameter, $G(x;\delta)$ and $g(x;\delta)$ are the input baseline CDF and PDF with a parameter vector δ.

Using the CDF and PDF of the traditional E distribution with scale parameter β as the inputs instead of $G(x;\delta)$ and $g(x;\delta)$ in (1) and (2), the two-parameter new distribution, namely unit half logistic geometric exponential (UHLGeE), can be constructed with the following CDF and PDF for $x\geq 0$ and $\theta,\beta>0$,

$$F(x;\theta,\beta)=\frac{2\left(1-e^{-\beta x}\right)}{\theta+(2-\theta)(1-e^{-\beta x})} \qquad (3)$$

$$f(x;\theta,\beta)=\frac{2\theta\beta e^{-\beta x}}{\left(\theta+(2-\theta)(1-e^{-\beta x})\right)^2} \qquad (4)$$

For more specialized cases, the CDF in (3) can be reformed according to the value of the shape parameter θ as follows

$$F(x;\theta,\beta)=\begin{cases}\dfrac{1-e^{-\beta x}}{1-\frac{2-\theta}{2}e^{-\beta x}} & ;\theta\in(0,2)\\[2ex] \dfrac{2(1-e^{-\beta x})}{\theta\left(1-\left(\frac{\theta-2}{\theta}\right)(1-e^{-\beta x})\right)} & ;\theta>2\\[2ex] 1-e^{-\beta x} & ;\theta=2\end{cases} \qquad (5)$$

The corresponding PDF will be

$$f(x;\theta,\beta)=\begin{cases}\dfrac{\theta\beta e^{-\beta x}}{2\left(1-\frac{2-\theta}{2}e^{-\beta x}\right)^2} & ;\theta\in(0,2)\\[2ex] \dfrac{2\beta e^{-\beta x}}{\theta\left(1-\left(\frac{\theta-2}{\theta}\right)(1-e^{-\beta x})\right)^2} & ;\theta>2\\[2ex] \beta e^{-\beta x} & ;\ \theta=2\end{cases} \qquad (6)$$

Equations (5) and (6) emphasize that the proposed UHLGeE distribution with $\theta=2$ is reduced to the baseline traditional E distribution, and thus it can also be described as a generalized E distribution.

The RF and HF can be attained as

$$R(x;\theta,\beta)=1-F(x;\theta,\beta)=\frac{\theta e^{-\beta x}}{\theta+(2-\theta)(1-e^{-\beta x})} \qquad (7)$$

$$h(x;\theta,\beta)=\frac{f(x;\theta,\beta)}{R(x;\theta,\beta)}=\frac{2\beta}{\theta+(2-\theta)(1-e^{-\beta x})} \qquad (8)$$

Figures 1 and 2 illustrate the versatility of the PDF and HF, demonstrating their potential to take different shapes. These graphical representations highlight the proposed distribution's adaptability and ability to capture various data patterns with increasing and decreasing HF.

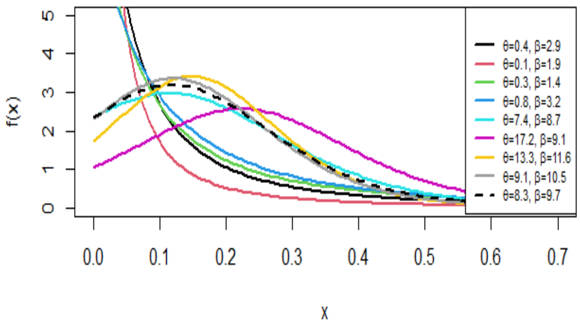

Figure 1. Plot of PDF with different default parameter values

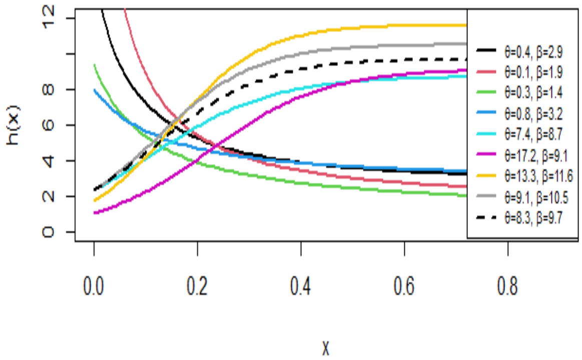

Figure 2. Plot of HF with different default parameter values

3. STATISTICAL FEATURES

This section investigates essential statistical features of the proposed distribution, including linear expansions of the PDF and CDF, moments, quantile measures, order statistics, and entropy. These features provide vital insights into the distribution's behavior and structure.

3.1. Linear expansions

Calling $F(x;\theta,\beta)$ from (5) and $f(x;\theta,\beta)$ from (6) with employing the following expansion series that valid for $|z|<1$ and $a>0, (1-z)^{-a}=\sum_{i=0}^{\infty}\binom{i+a-1}{i}z^i$ and $(1-z)^a=\sum_{i=0}^{\infty}(-1)^i\binom{a}{i}z^i$, the linear expansions of the CDF and PDF formulae will be

$$F^e(x;\theta,\beta)=\begin{cases}\sum_{i,j=0}^{\infty}c_{i,j}\,(1-e^{-\beta x})^{j+1};\theta\in(0,2)\\ \frac{2}{\theta}\sum_{i=0}^{\infty}u_i\,(1-e^{-\beta x})^{i+1}\,;\theta>2\end{cases} \quad (9)$$

and

$$f^e(x;\theta,\beta)=\begin{cases}\frac{\theta}{2}\beta e^{-\beta x}\sum_{i,j=0}^{\infty}c_{i,j}\,(i+1)\\ \qquad (1-e^{-\beta x})^j\,;\theta\in(0,2)\\ \frac{2}{\theta}\beta e^{-\beta x}\sum_{i=0}^{\infty}u_i(i+1)\\ \qquad (1-e^{-\beta x})^i\,;\theta>2\end{cases} \quad (10)$$

where $c_{i,j}=c_{i,j}(\theta)=(-1)^j\left(\frac{2-\theta}{2}\right)^i\binom{i}{j}$, $\binom{i}{j}=\frac{i!}{j!(i-j)!}$, and $u_i=u_i(\theta)=\left(\frac{\theta-2}{\theta}\right)^i$.

It should be noted that, when $\theta=2$, the CDF expansion formula and its associated PDF are respectively related to the expansion CDF and PDF of the baseline E distribution.

3.2. Quantile Measurements

The quantile function $Q(q)$of the proposed distribution can be obtained easily by solving (3) for $Q(q)$ in terms of q, i.e., $F(Q(q);\theta,\beta)=q$, as

$$Q(q)=x_q=\frac{-1}{\beta}\ln\left(\frac{2(1-q)}{2-(2-\theta)q}\right);q\in(0,1),\theta>0 \quad (11)$$

By replacing q with u, where u denotes a value of standard Uniform distribution, simulate a random variable related to the UHLGeE is

$$Q(u)=x_q=\frac{-1}{\beta}\ln\left(\frac{2(1-u)}{2-(2-\theta)u}\right);u\in(0,1),\theta>0 \quad (12)$$

With specific values of q, see Al-Noor & Hadi (2021) related quantile measures can be attained, including the median by setting $q=1/2$, skewness $\left(S(X)=\frac{Q(3/4)-2Q(2/4)+Q(1/4)}{Q(3/4)-Q(1/4)}\right)$, and kurtosis $\left(K(X)=\frac{Q(7/8)-Q(5/8)-Q(3/8)+Q(1/8)}{Q(6/)-Q(2/8)}\right)$.

3.3. Order Statistics (O.S.)

Consider $X_{1:n}\leq\ldots\leq X_{n:n}$ denote the O.S. related to a random sample of size n related to UHLGeE. The PDF of the i[th] O.S. can be expressed for $i=1,2,\ldots,n$ as

$$f_{i:n}(x;\theta,\beta)=\frac{n!2^i\theta^{n-i+1}\beta\left(1-e^{-\beta x}\right)^{i-1}e^{-\beta x(n-i+1)}}{(i-1)!(n-i)!\left(\theta+(2-\theta)\left(1-e^{-\beta x}\right)\right)^{n+1}} \quad (13)$$

The PDFs of the smallest and largest O.S. can be attained simply from (13) with $(i=1)$ and $(i=n)$ respectively, as

$$f_{1:n}(x;\theta,\beta)=\frac{2n\theta^n\beta e^{-n\beta x}}{\left(\theta+(2-\theta)(1-e^{-\beta x})\right)^{n+1}} \quad (14)$$

$$f_{n:n}(x;\theta,\beta)=\frac{2^n n\theta\beta e^{-\beta x}\left(1-e^{-\beta x}\right)^{n-1}}{\left(\theta+(2-\theta)(1-e^{-\beta x})\right)^{n+1}} \quad (15)$$

3.4. Moments and Moments Generating Function

Using the definition of the r^{th} moment about the origin, $\mu_r=E(X^r)=\int_0^{\infty}x^r f(x;\theta,\beta)\,dx$ with the expansion PDF given in (10), the outcomes are easily attained as

$$\mu_r=E(X^r)=\begin{cases}\frac{\theta}{2}\sum_{i,j=0}^{\infty}c_{i,j}\,(i+1)\,\mu_{r,j}^{(PW)}\ ;\ \theta\in(0,2)\\ \frac{2}{\theta}\sum_{i=0}^{\infty}u_i\,(i+1)\,\mu_{r,i}^{(PW)}\ ;\ \theta>2\end{cases} \quad (16)$$

where the probability weighted moments

$$\mu_{r,j}^{(PW)}=\sum_{k=0}^{\infty}(-1)^k\binom{j}{k}\frac{1}{k+1}\int_0^{\infty}x^r\,\beta(k+1)e^{-\beta(k+1)x}dx$$

and

$$\mu_{r,i}^{(PW)}=\sum_{j=0}^{\infty}(-1)^j\binom{i}{j}\frac{1}{j+1}\int_0^{\infty}x^r\,\beta(j+1)e^{-\beta(j+1)x}dx.$$

Since the two integrals represent the r^{th} moments about the origin of the traditional E distribution respectively with parameters $(\beta(k+1))$ and $(\beta(j+1))$, then

$$\mu_{r,j}^{(PW)}=\sum_{k=0}^{\infty}(-1)^k\binom{j}{k}\frac{r!}{\beta^r(k+1)^{r+1}} \quad (17)$$

and

$$\mu_{r,i}^{(PW)}=\sum_{j=0}^{\infty}(-1)^j\binom{i}{j}\frac{r!}{\beta^r(j+1)^{r+1}} \quad (18)$$

Substituting (17) and (18) in (16), the r^{th} moment of UHLGeE is

$$\mu_r = \begin{cases} \frac{\theta}{2}\sum_{i,j,k=0}^{\infty} c_{i,j}\,(-1)^k \binom{j}{k}\frac{r!\,(i+1)}{\beta^r(k+1)^{r+1}}; \theta \in (0,2) \\ \frac{2}{\theta}\sum_{i,j=0}^{\infty} u_i\,(-1)^j \binom{i}{j}\frac{r!\,(i+1)}{\beta^r(j+1)^{r+1}} \quad ; \theta > 2 \end{cases} \quad (19)$$

The moments generating function of the UHLGeE can be attained by applying its definition with the Maclaurin series by $M_X(t) = E(e^{tX}) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r$, and μ_r given in (2.21), as

$$M_X(t) = \begin{cases} \frac{\theta}{2}\sum_{i,j,k,r=0}^{\infty} c_{i,j}\binom{j}{k}\frac{(-1)^k\,t^r\,(i+1)}{\beta^r(k+1)^{r+1}}; \theta \in (0,2) \\ \frac{2}{\theta}\sum_{i,j,r=0}^{\infty} u_i\binom{i}{j}\frac{(-1)^j\,t^r(i+1)}{\beta^r(j+1)^{r+1}} \quad ; \theta > 2 \end{cases} \quad (20)$$

Further, the characteristic function can also be attained by the mentioned series as $\emptyset_X(t) = E(e^{itX}) = \sum_{r=0}^{\infty}\frac{(it)^r}{r!}\mu_r$.

3.5. Entropies

According to information theory, a random variable's entropy measures the typical degree of information or ambiguity surrounding its various states or outcomes. Rényi and Shannon, two major types of entropy, are discussed due to their important applications. For more details about these entropies, including their formulas, see Reyad et al., 2018 and Al-Noor & Sultan, 2024.

The Rényi entropy can be specified based on the PDF in (4) by

$$I_R(\Upsilon) = \frac{1}{1-\Upsilon}\ln\left(\int_{\forall x} f^{\Upsilon}(x;\theta,\beta)\,dx\right); \Upsilon > 0, \Upsilon \neq 1.$$

Then, using the generalized binomial formula, $f^{\Upsilon}(x;\theta,\beta)$ can be expressed as a series expansion, which is then applied to get the formula of $I_R(\Upsilon)$ as

$$I_R(\Upsilon) = \begin{cases} \frac{1}{1-\Upsilon}\left[\Upsilon\ln\left(\frac{\theta}{2}\right) + \ln(A_1)\right] ; \theta \in (0,2) \\ \frac{1}{1-\Upsilon}\left[\Upsilon\ln\left(\frac{2}{\theta}\right) + \ln(A_2)\right] ; \theta > 2 \end{cases} \quad (21)$$

where

$$A_1 = \sum_{i,j,k=0}^{\infty} c_{i,j}(\theta,\Upsilon)\,(-1)^k\binom{j}{k}\frac{\beta^{\Upsilon-1}}{\gamma+k};$$

$$A_2 = \sum_{i,j=0}^{\infty} u_i(\theta,\Upsilon)\,(-1)^j\binom{i}{j}\frac{\beta^{\Upsilon-1}}{\gamma+j};$$

$c_{i,j}(\theta,\Upsilon) = (-1)^j\left(\frac{2-\theta}{2}\right)^i\binom{i}{j}\binom{2\Upsilon+i-1}{i} = c_{i,j}\binom{2\Upsilon+i-1}{i}$; and $u_i(\theta,\Upsilon) = \left(\frac{\theta-2}{\theta}\right)^i\binom{2\Upsilon+i-1}{i} = u_i\binom{2\Upsilon+i-1}{i}$.

The Shannon entropy can be obtained by applying $\Upsilon \longrightarrow 1$ to $I_R(\Upsilon)$ in (21) or is specified by $\eta(X) = E\big(-\ln(f(x;\theta,\delta))\big)$. Recall (4), then

$$\eta(X) = \begin{cases} \ln\left(\frac{2}{\theta}\right) - E\left(\ln\left(\beta e^{-\beta X}\right)\right) + 2E_1 \;; \theta \in (0,2) \\ \ln\left(\frac{\theta}{2}\right) - E\left(\ln\left(\beta e^{-\beta X}\right)\right) + 2E_2 \;; \theta > 2 \end{cases} \quad (22)$$

where $E\left(\ln\left(\beta e^{-\beta X}\right)\right) = \ln(\beta) - \beta E(X)$, and the other required mathematical expectations can be obtained with $\ln(1-z) = -\sum_{i=1}^{\infty}\frac{1}{i}z^i; |z| < 1$, along with the expansion exponential series, and $(1-z)^a$ as

$$E_1 = E\left(\ln\left(1 - \frac{2-\theta}{2}e^{-\beta X}\right)\right)$$
$$= -\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\frac{1}{i}c_{i,j}\;E\left(\left(1-e^{-\beta X}\right)^j\right)$$
$$= \sum_{i=1}^{\infty}\sum_{j,k,r=0}^{\infty} c_{i,j}\frac{(-1)^{k+r+1}}{r!\,i}\binom{j}{k}(k\beta)^r E(X^r)$$

and

$$E_2 = E\left(\ln\left(1 - \frac{\theta-2}{\theta}\left(1-e^{-\beta X}\right)\right)\right)$$
$$= -\sum_{i=1}^{\infty}\frac{1}{i}u_i\,E\left(\left(1-e^{-\beta X}\right)^i\right)$$
$$= \sum_{i=1}^{\infty}\sum_{j,r=0}^{\infty} u_i\frac{(-1)^{j+r+1}}{r!\,i}\binom{i}{j}(j\beta)^r E(X^r)$$

where $c_{i,j}$ and u_i as defined previously, and $E(X)$ can be attained from $E(X^r)$ given in (19) with $r = 1$.

4. INFERENCE WITH DIFFERENT METHODS

For this purpose, consider $x_1, x_2, \dots, x_n$ a random sample of size n from the UHLGeE represents the data, and their ascending ordering values are denoted by $x_{(1)}, x_{(2)}, \dots, x_{(n)}$. Four methods: maximum likelihood (ML), ordinary least squares (OLS), weighted least squares (WLS), and Cramer Von-Mises (CVM) minimum distance are considered. For more details on the considered methods, see Hassan et al., 2023.

The ML methodology is the most often used parameter estimation method due to its desirable features such as consistency, asymptotic efficiency, and invariance (Dey et al., 2019). The ML estimates of a vector of parameters $\varphi = \theta, \beta$ are attained through $\hat{\varphi}^{\text{ML}} = \operatorname{argmax} \mathrm{L}(\varphi)$ or, equivalently $\hat{\varphi}^{\text{ML}} = \operatorname{argmax} \ell(\varphi)$ where $\mathrm{L}(\varphi) = \prod_{i=1}^{n} f(x_i;\varphi)$ and $\ell(\varphi) = \ln(\mathrm{L}(\varphi))$ are the likelihood and natural logarithm likelihood functions. Now, recall the PDF in (4), then

$$\ell(\theta,\beta) = n\ln(2) + n\ln(\theta) + n\ln(\beta) - \beta\sum_{i=1}^{n} x_i - 2\sum_{i=1}^{n}\ln\left(\theta + (2-\theta)(1-e^{-\beta x_i})\right)$$

The ML estimates of the parameters can be obtained by solving simultaneously the two non-linear equations $\partial\ell(\theta,\beta)/\partial\theta = 0$ and $\partial\ell(\theta,\beta)/\partial\beta = 0$, where

$$\frac{\partial\ell(\theta,\beta)}{\partial\theta} = \frac{n}{\theta} - 2\sum_{i=1}^{n}\frac{e^{-\beta x_i}}{\theta + (2-\theta)(1-e^{-\beta x_i})} \quad (23)$$

and

$$\frac{\partial \ell(\theta,\beta)}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n} x_i - 2(2-\theta) \sum_{i=1}^{n} \frac{x_i\, e^{-\beta x_i}}{\theta + (2-\theta)(1-e^{-\beta x_i})} \quad (24)$$

On the other side, the OLS estimates of φ are attained through $\hat{\varphi}^{\text{OLS}} = \operatorname{argmin} \text{OL}S(\varphi)$ where $\text{OL}S(\varphi) = \sum_{i=1}^{n} \left(F(x_{(i)};\varphi) - \frac{i}{n+1} \right)^2$ and $F(x_{(i)};\varphi)$ is the CDF of the considered distribution with ascending ordering observations. Now, recall the CDF given in (3), then

$$\text{OLS}(\theta,\beta) = \sum_{i=1}^{n} \left(\frac{2(1-e^{-\beta x_{(i)}})}{\theta + (2-\theta)(1-e^{-\beta x_{(i)}})} - \frac{i}{n+1} \right)^2 \quad (25)$$

Similarly, the WLS estimates of φ are attained through $\hat{\varphi}^{\text{WLS}} = \operatorname{argmin} \text{WL}S(\varphi)$ where $\text{WL}S(\varphi) = \sum_{i=1}^{n} w_i \left(F(x_{(i)};\varphi) - \frac{i}{n+1} \right)^2$ and $w_i = \frac{(n+1)^2(n+2)}{i(n-i+1)}$. Based on the CDF given in (3), then

$$\text{WLS}(\theta,\beta) = \sum_{i=1}^{n} w_i \left(\frac{2(1-e^{-\beta x_{(i)}})}{\theta + (2-\theta)(1-e^{-\beta x_{(i)}})} - \frac{i}{n+1} \right)^2 \quad (26)$$

Finally, the CVM estimates of φ are attained through $\hat{\varphi}^{\text{CVM}} = \operatorname{argmin} \text{CVM}(\varphi)$ where $\text{CVM}(\varphi) = \frac{1}{12n} + \sum_{i=1}^{n} \left(F(x_{(i)};\varphi) - \frac{2i-1}{2n} \right)^2$. Based on the CDF given in (3), then

$$\text{CVM}(\theta,\beta) = \frac{1}{12n} + \sum_{i=1}^{n} \left(\frac{2(1-e^{-\beta x_{(i)}})}{\theta + (2-\theta)(1-e^{-\beta x_{(i)}})} - \frac{2i-1}{2n} \right)^2 \quad (27)$$

Based on (25), (26), and (27), the OLS, WLS, and CVM estimates can be obtained by solving simultaneously the two non-linear equations attained from taking the partial derivatives of $\text{OLS}(\theta,\beta)$, $\text{WLS}(\theta,\beta)$, and $\text{CVM}(\theta,\beta)$ concerning the unknown parameters (θ,β) and equating them to zero.

5. EMPIRICAL AND REAL APPLICATIONS

To evaluate the model's behavior concerning parameter estimations over different default values and sample sizes, simulation experiments are conducted. Furthermore, the utility of the proposed distributions with various statistical measures is evaluated through a real dataset. All graphical and numerical illustrations are obtained using the R programming language.

5.1. Empirical Application

The performances of the discussed estimation methods are investigated empirically via simulation. Based on different default parameter sets and sample sizes, simulation studies experimentally evaluate the behavior of the ML, OLS, WLS, and CVM estimates for parameters. The default parameter values are set as: set I $(\theta = 0.4, \beta = 2.9)$, set II $(\theta = 17.2, \beta = 9.1)$, and set III $(\theta = 13.3, \beta = 11.6)$. The sample sizes are chosen as: $n = 25, 50, 100$, and 200. The simulated formula in (12) is used to generate a random sample of size n that follows UHLGeE. This step is repeated 1000 times to yield 1000 independent samples of varying sizes. After obtaining the ML, OLS, WLS, and CVM estimates, calculate the mean squared error (MSE) (Al-Noor & Abd Al-Ameer, 2013) for each parameter, as follows

$$MSE(\hat{\varphi}) = \frac{1}{1000} \sum_{i=1}^{1000} (\hat{\varphi}_{\text{i}} - \varphi)^2 ;\ \varphi = \theta, \beta \quad (28)$$

Tables 1 – 3 present the empirical results of the simulation studies of the considered set. Further, Table 4 presents the number of times that a specified method app ears as the best method.

TABLE 1. Values of MSE of different estimates for Set I

φ	n	ML	OLS	WLS	CVM	Best
θ	25	0.025163	0.040717	0.033419	0.055569	ML
	50	0.012423	0.019829	0.017620	0.023257	ML
	100	0.007168	0.011044	0.008964	0.012039	ML
	200	0.004523	0.007122	0.005720	0.007371	ML
β	25	0.106065	0.150210	0.128624	0.187161	ML
	50	0.064048	0.104979	0.089875	0.115435	ML
	100	0.039350	0.063872	0.050265	0.067242	ML
	200	0.025820	0.043173	0.033616	0.043773	ML

TABLE 2. Values of MSE of different estimates for Set II

φ	n	ML	OLS	WLS	CVM	Best
θ	25	0.891148	1.014250	1.048576	1.491813	ML
	50	0.407929	0.415534	0.406135	0.497186	WLS
	100	0.241490	0.247699	0.242495	0.269948	ML
	200	0.148502	0.155783	0.150715	0.161957	ML
β	25	0.074956	0.079702	0.075895	0.086233	ML
	50	0.047855	0.053458	0.050117	0.055340	ML
	100	0.034801	0.038384	0.036215	0.039055	ML
	200	0.024134	0.027136	0.025184	0.027198	ML

TABLE 3. Values of MSE of different estimates for Set III

φ	n	ML	OLS	WLS	CVM	Best
θ	25	0.714291	0.667275	0.642940	0.951644	WLS
	50	0.309732	0.355257	0.330148	0.419651	ML
	100	0.175848	0.178639	0.171925	0.195199	WLS
	200	0.105318	0.111173	0.106752	0.114202	ML
β	25	0.103398	0.111535	0.107009	0.122748	ML
	50	0.066509	0.074938	0.069464	0.077397	ML
	100	0.045873	0.051771	0.047708	0.053029	ML
	200	0.030769	0.035552	0.032405	0.035389	ML

TABLE 4. Number of times that the specified estimates is the best

Set	ML	OLS	WLS	CVM
I	8	0	0	0
II	7	0	1	0
III	6	0	2	0

From the empirical results in Tables 1–3, the most important outcomes can be stated as:

1. The ML method is the best for estimating parameter θ for all sample sizes in set I as well as in other subsequent sets, except II with $n = 50$, and III with $n = 25, 100$, where the WLS method is the best.
2. The ML method outperformed other methods in estimating parameter β across all sets and sample sizes.
3. Overall, for both parameters, the ML method was most often reported as the best. Noting that neither OLS nor CVM recorded any appearances. See Table (4).
4. The MSE values decrease with increasing sample size, demonstrating the estimators' consistency.

5.2. Real Application

The dataset used to examine the applicability of the proposed distribution reflects the birth rate with abnormalities in Iraqi governorates in 2020, as reported in Iraq's Ministry of Health Annual Report. The values of the considered data are: "2.9, 2.0, 3.8, 2.1, 1.8, 1.6, 10.9, 4.0, 5.8, 3.9, 2.5, 3.6, 2.0, 0.2, 3.2, 1.1, 3.2, 1.2". The basic descriptive statistics for this medical dataset are given in Table 5. The common statistical information criteria (IC) are employed for the evaluation process, as tools for model selection, Akaike: $A = 2(m - \hat{\ell})$; Consistent Akaike: $CA = 2\left(\frac{nm}{n-m-1} - \hat{\ell}\right)$; Bayesian: $B = m\ln(n) - 2\hat{\ell}$; and Hannan and Quinn: $HQ = 2\big(m\ln(\ln(n)) - \hat{\ell}\big)$ where n: sample size, m: number of estimated parameters, and $\hat{\ell}$: estimated ℓ evaluated at ML estimates, as their performances are quite satisfactory for all competitive distributions. The p-values of the KS (Kolmogorov-Smirnov) goodness-of-fit statistic are also considered during the evaluation procedure. The model with a smaller value of the IC and a higher p-value is preferred (Alizadeh et al., 2018). To broaden the investigation process, the traditional E along with its common extended versions, beta E (BE), Kumaraswamy E (KE), exponentiated generalized E (EGE), Weibull E (WE), and Gompertz E (GoE), are also included. For more details about the families to which the competing distributions belong, see Eugene et al. (2002), Cordeiro & De Castro (2011), Cordeiro et al. (2013), Bourguignon et al. (2014), and Alizadeh et al. (2017). The numerical results are displayed in Tables 6 and 7.

TABLE 5. Basic descriptive statistics for Data

St.	Mean	Sk	Ku	Q1	Med.	Q3
Val.	3.1	2.27	6.96	1.75	2.7	3.825

TABLE 6. ML estimate's values for Data

Dist.	$\hat{\delta}$	$\hat{\theta}$	$\hat{\beta}$
UHLGeE		11.94395	0.72182
BE	1.08121	2.23714	0.48139
KE	1.56019	2.07478	0.36064
EGE	2.25284	1.14394	0.44959
WE	1.30352	1.46570	0.37939
GoE	0.32812	0.74871	0.32290
E			0.32258

TABLE 7. Values of fitting Data

Dist.	A	CA	B	HQ	KS p-value
UHLGeE	76.4194	77.2194	78.2001	76.6649	0.8160
BE	78.1144	79.8287	80.7855	78.4827	0.7855
KE	78.0645	79.7788	80.7356	78.4328	0.7776
EGE	78.1156	79.8299	80.7868	78.4840	0.7866
WE	78.7679	80.4822	81.4390	79.1362	0.6363
GoE	81.2902	83.0045	83.9613	81.6585	0.5328
E	78.7304	78.9804	79.6208	78.8532	0.2376

The numerical outcomes indicate that: The descriptive statistics show that the data set is right-skewed and leptokurtic, The proposed UHLGeE has the lowest values of all IC and the largest p-values compared to other competitive distributions, making it the most appropriate distribution to represent the considered medical data. Further, the p-values for all distributions are significant, higher than 0.05. Thus, all distributions are fit for modeling data, but the proposed one is the best. Moreover, the best fitting of UHLGeE is shown by the plots of the estimated PDFs and CDFs in Figures 3 and 4.

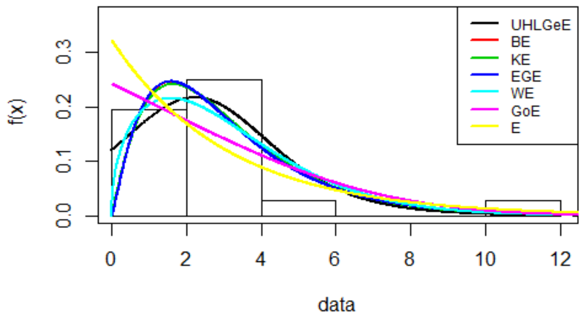

Figure 3. Estimated PDF of UHLGeE and competitive distributions

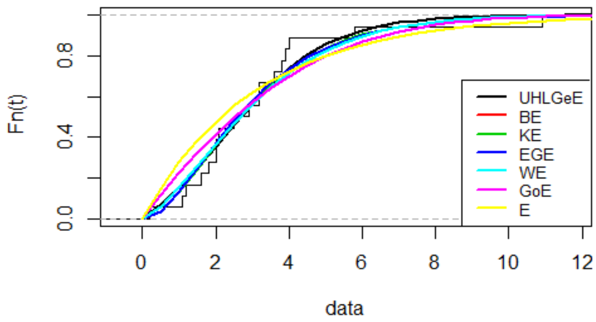

Figure 4. Empirical CDF of UHLGeE and competitive distributions

6. CONCLUSIONS

Over the years, there has been renewed interest in constructing new univariate continuous distributions. In this paper, the unit half logistic geometric exponential (UHLGeE) distribution is proposed as a new flexible extension of the traditional exponential (E) distribution. The proposed distribution includes incorporating one extra parameter into the existing traditional E distribution. The traditional version can be obtained as a special case of the proposed distribution. Thus, the proposed distribution can be characterized as a generalized exponential distribution. The UHLGeE's density and hazard functions can have a variety of shapes, indicating its applicability to analyzing varied real-life data. Further, the closed expression of the quantile function is what makes the proposed distribution remarkable. Inverse transform sampling can therefore be used to provide the distribution values, and skewness and kurtosis measurements are simple to compute. Through a wide range of simulation experiments, the accuracy and stability of the two parameters are emphasized by employing four estimation techniques, and then providing optimism regarding the distribution's flexibility in real-life applications. The proposed distribution has the lowest values of all information criteria and the largest p-values compared to other competitive distributions, making it the most appropriate distribution to represent the considered real-life data. For future studies, it is recommended to adopt and employ UHLGeE in additional applications due to its features and adaptability, and consider rank set sampling or Bayesian methods besides the four methods considered here to estimate the unknown parameters and reliability measures.

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