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New Weighted Rayleigh Version of Azzalini Distribution

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A b s t r a c t

In this article a new weighted Rayleigh distribution has been proposed. The main difference between the proposed distribution and the other existing Rayleigh families is that, usually, when families add such a parameter, it creates new behavior, such as (tail). In the proposed distribution, the new parameter η interacts with β to adjust the spreading. It is simpler and more computationally.
The distribution's properties have been analyzed, and the maximum-likelihood method was suggested for the estimation of the parameters of the new distribution. The novel weighted distribution was exemplified through its application to an actual lifetime dataset.

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1. INTRODUCTION

Statistical distributions serve to represent sample data gathered from a population or to characterize the results of a random experiment.
Classical distributions are not always sufficient for modeling real data. In most applied fields, such as lifetime analysis, there is a very strong need for extended forms of the classical distributions. In the literature, the Rayleigh distribution has received close attention due to the benefits it offers in comparison to other distributions in the modeling of lifetime data. lord. Rayleigh, in 1880, introduced a one-scale parameter distribution called the Rayleigh distribution and considered it one of the most widely used distributions. The researchers developed various generalizations of this distribution to enhance its flexibility in modeling lifetime data.
(Azzalini, 1985). Azzalini in 1985 , proposed a new class of probability distributions parameterized by the shape parameter (Gupta & Kundu, 2009). Based on Azzalini's idea, Gupta and kundu in 2009 introduced a shape parameter for the exponential distribution.
(Nasiru, 2015). In 2015,Nasiru and Suleman proposed a new distribution based on Azzalini's (1985) approach; its mathematical properties derived, and its applicability demonstrated on real data (Mahmood A., 2020). In 2020, Mahmood A. Sahmran, proposed a weighted distribution based on the conventional type I Pareto distribution (Ajami & Jahanshahi, 2016). In 2016, Ajami and Jahanshahi made a comparison of different methods of estimation for the size-biased Rayleigh distribution (Bashir & Rasul, 2018). In 2018, Bashir and Rasul proposed the Rayleigh distribution that is biased by area (ARD) with derived properties and parameter estimates, showing a good fit to lifetime data using statistical tests (Bhat & Ahmad, 2020). Bhat and Ahmad in 2020, proposed the Power Rayleigh distribution, studied its properties and entropies, by using MLE estimates of parameters, and then demonstrated its usefulness with real data (Ahad et al., 2021). Ahad, Ahmad, and Rehman in 2021, after comparing the two estimation methods of the Bayesian and non-Bayesian weighted Rayleigh distribution's parameters, found that the Bayesian method under entropy loss with a Gumbel Type II prior is the most effective (Hussein et al., 2023). In 2023, Lamyaa , Huda, and Iden enhanced the ER and MWER distributions, derived the statistical properties, and, for greater modeling flexibility, showed that MWER generalizes ER. (Ilori et al., 2025). In 2025, Ilori, Adetunji , Awogbemi, Damilare, Toyosi and Adebisi proposed the Weighted Two-Parameter Rayleigh (W2R) distribution, an extension of the

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Rayleigh distribution using an inverted weight function with an extra parameter. This new model improves flexibility for reliability and survival analysis.

This article proposes an innovative extension of the Rayleigh distribution, termed the "new weighted Rayleigh version of Azzalini's distribution," predicated on a modified weighted variant of Azzalini's (1985) work. Additionally, the principal statistical characteristics of the newly modified distribution were proposed. The conclusion of this paper has been established utilizing a literature dataset.

2. MATERIAL AND METHODS

2-1. NEW WEIGHTED RAYLEIGH DISTRIBUTION

In this part of the article, the probability density function of the new weighted Rayleigh Distribution has been derived according to the The Gupta and Kunda formula, equation (1), and Azzalini's concept. The year 2009, Gupta and Kundu introduced a novel class of Weighted Exponential distributions by utilizing the approach proposed by Azzalini (1985) on the exponential distribution.

$$f_w(x;\beta,\eta) = \frac{w(x).f(x)}{[E(w(x))]} \qquad (1)$$

Where

$w(x)$ represents weight function, (where must satisfy two conditions to ensure that the resulting pdf integrates to one: Non- negativity and Normalization) +

$f(x)$ is represented as a probability density function

$E(w(x))$ serves as a normalized

A continuous random variable that is not negative x is defined as possessing a one-parameter Rayleigh distribution, described by a (pdf) of the following form:

$$f(x;\beta) = \frac{x}{\beta^2}\, e^{-\left(\frac{x^2}{2\beta^2}\right)} \quad ; x \geq 0 \,, \beta > 0 \qquad (2)$$

The (CDF) of the Rayleigh distribution is expressed as:

$$F(x;\beta) = 1 - e^{-\left(\frac{x^2}{2\beta^2}\right)} \quad ; x \geq 0 \,, \beta > 0 \qquad (3)$$

The Survival of Rayleigh distribution function is provided in the form:

$$S(t;\beta) = e^{-\left(\frac{t^2}{2\beta^2}\right)} \quad ; t \geq 0 \,, \beta > 0 \qquad (4)$$

Now from the equation (4) ; Let $w(x) = \left(s(x)\right)^{\eta}$

$$w(x) = \left[e^{-\left(\frac{x^2}{2\beta^2}\right)}\right]^{\eta} = e^{-\left(\frac{\eta x^2}{2\beta^2}\right)} \qquad (5)$$

Now, by putting equations (2) and (5) in equation (1), and the normalized A = $1+\eta$ we get the probability density function of the new weighted Rayleigh distribution, which is defined as:

$$f_w(x;\beta,\eta) = (1+\eta)\frac{x}{\beta^2}.e^{-\frac{(\eta+1)\,x^2}{2\beta^2}} \; ; \; x > 0 \,, \beta > 0 \,, \eta > -1 \qquad (6)$$

β represents a scale parameter, while η denotes a new shape parameter.

<u>Lemma1</u>

The new weighted Rayleigh distribution

$f_w(x;\beta,\eta) = (1+\eta)\frac{x}{\beta^2}.e^{-\frac{(\eta+1)\,x^2}{2\beta^2}} \; ; \; x > 0 \,, \beta > 0 \,, \eta > -1$ It is a probability density function.

<u>Proof</u>

$$\int_0^{\infty} f_w(x;\beta,\eta)\, dx = \int_0^{\infty} \frac{(1+\eta)x\, e^{-\frac{(\eta+1)\,x^2}{2\beta^2}}}{\beta^2}\, dx \qquad (7)$$

$$\text{Let } \; u = -\frac{x^2(1+\eta)}{2\beta^2} \qquad (8)$$

$$du = -\frac{x(1+\eta)}{\beta^2}\, dx \qquad (9)$$

Put equations (8) and (9) in (7) we get:

$$= -\int_0^{\infty} e^u\, du \qquad (10)$$

Apply the exponential rule:

$\int_0^{\infty} a^u\, du = \frac{a^u}{\ln(a)}$, with a = e = e^u

Put it in equation (10) , $= -e^u$

Now, undo substitutes the value of u from the equation (8), we get:

$$= -e^{-\frac{x^2(1+\eta)}{2\beta^2}} + c$$

$$= \frac{1+\eta}{2\left(\frac{\eta}{2\beta^2}+\frac{1}{2\beta^2}\right)\beta^2} = 1$$

Then, $f_w(x;\beta,\eta) = (1+\eta)\frac{x}{\beta^2}.e^{-\frac{(\eta+1)\,x^2}{2\beta^2}} , x > 0 \,, \beta > 0 \,, \eta > -1$,Is a p.d.f weighted Rayleigh distribution.The proposed distribution is a new method to explore the Rayleigh distribution, not essentially a new geometric shape. The new model supports the example in over-parameterized models. The cumulative distribution function (cdf) of the new weighted Rayleigh distribution is expressed as follows:

$$F_w(x;\beta,\eta) = 1 - e^{-\frac{(\eta+1)\,x^2}{2\beta^2}} \qquad (11)$$

The above CDF is considered novel because the proposed model can be considered a kind of sensitive analysis In other words, it can be written in equation of simple way (sometimes called closed forms) while most models are difficult to solve by hand.

Figures (1) and (2) illustrate the pdf and cdf , respectively, of the novel weighted Rayleigh distribution for specific values of the parameter of the scale (β) and the new parameter of shape (η).

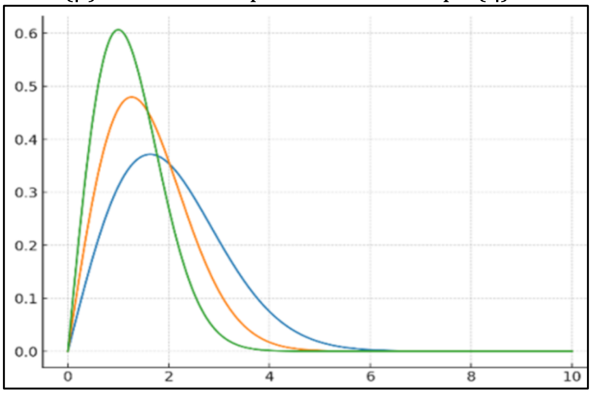

Figure 1. illustrates PDF of the new weighted Rayleigh distribution for β = 2.0 and η = 0.5, 1.5, 3.0.

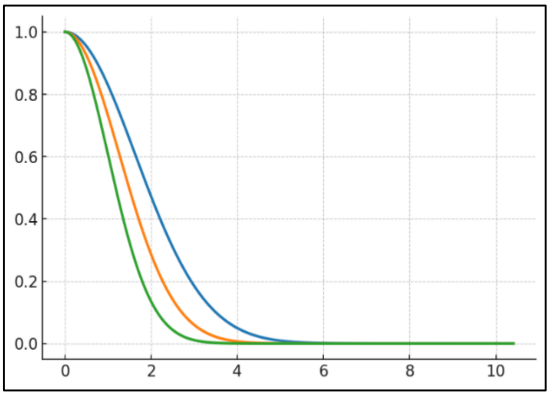

Figure 2. illustrates the CDF of the new weighted Rayleigh distribution with β = 2.0 and η values of 0.5, 1.5, and 3.0.

The new weighted Rayleigh distribution's survival function is:

$$S(x) = e^{-\frac{(\eta+1)\,x^2}{2\beta^2}} \qquad (12)$$

In terms of the hazard function, the new weighted Rayleigh distribution has been obtained from equations (6) and (12) by:

$$\mathrm{H(x)} = \frac{f(x)}{s(x)} = \frac{(1+\eta)\frac{x}{\beta^2}.e^{-\frac{(\eta+1)\,x^2}{2\beta^2}}}{e^{-\frac{(\eta+1)\,x^2}{2\beta^2}}}$$

$$\mathrm{H(x)} = (1+\eta)\,\frac{x}{\beta^2} \qquad (13)$$

Figure 3 illustrates the hazard function of the new weighted Rayleigh distribution for different selected values for the scale parameter (β) and the new shape parameter (η).

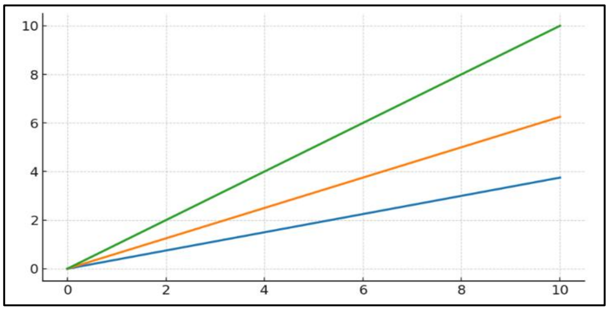

Figure 3. illustrates the hazard function of the new weighted Rayleigh distribution with β = 2.0 and η = 0.5, 1.5, 3.0.

The values of parameters in proposed model are selected to serve as representative standards. They chosen to provide a sensitive wide-ranging analysis, typical of how the distribution acts under changing steps of weighting intensity (such as high bias).

2-2. MOST IMPORTANT STATISTICAL PROPERTIES OF THE NWRD

An examination of the most significant statistical characteristics of the newly developed weighted Rayleigh distribution has been carried out here: the r-th raw moment, the mean, the variance, the moment generating function, the factorial moment-generating function, the mode, the median, the coefficient of skewness, the coefficient of kurtosis, the quantile function, and the characteristic function.

2-2-1. THE MOMENTS

Lemma2

Let x to be a r.v. of the NWRD, then the r^{th} moment function of the NWRD is $-\frac{2^{\frac{r}{2}}\,\Gamma\left(\frac{r+2}{2},\frac{(\eta+1)x^2}{2\beta^2}\right)|\beta|^r\,x^r}{(\eta+1)^{\frac{r}{2}}\,|x|^r}$.

All higher-order moments for the proposed distribution are finite for all effective parameter values.

proof

The r-th moment about the origin of the new weighted Rayleigh distribution is:

$$E(x^r) = \int_0^{\infty} x^r (1+\eta)\frac{x}{\beta^2}.e^{-\frac{(\eta+1)\,x^2}{2\beta^2}}dx$$

$$E(x^r) = \frac{(1+\eta)\,x^{r+1}.e^{-\frac{(\eta+1)\,x^2}{2\beta^2}}}{\beta^2}\,dx \quad (14)$$

Now, let

$$u = \frac{\beta^{-r-2}\,(\eta+1)^{\frac{r+2}{2}}\,x^{r+2}}{2^{\frac{r+2}{2}}} \quad (15)$$

$$du = \frac{(r+2)\,\beta^{-r-2}\,(\eta+1)^{\frac{r+2}{2}}\,x^{r+1}}{2^{\frac{r+2}{2}}}\,\mathrm{dx} \quad (16)$$

Put the equations (15) , (16) in (14)

$$= \frac{2^{\frac{r}{2}+1}\,\beta^r\,(\eta+1)^{1-\frac{r}{2}}}{(r+2)\eta+r+2}\int_0^{\infty} e^{-u^{\frac{2}{r+2}}}du \quad (17)$$

Now, to solve the above integral from equation (17)

This is a special integral (incomplete gamma function)

$$= -\frac{(r+2)\,\Gamma\left(\frac{r+2}{2},u^{\frac{2}{r+2}}\right)}{2} \quad (18)$$

Plug equation (18) in equation (14)

$$= -\frac{(r+2)\,2^{\frac{r}{2}}\,\beta^r\,(\eta+1)^{1-\frac{r}{2}}\,\Gamma\left(\frac{r+2}{2},u^{\frac{2}{r+2}}\right)}{(r+2)\eta+r+2}$$

Undo substitute the value of u

$$= -\frac{(r+2)\,2^{\frac{r}{2}}\,\beta^r\,(\eta+1)^{1-\frac{r}{2}}\,\Gamma\left(\frac{r+2}{2},u^{\frac{2}{r+2}}\right)}{(r+2)\eta+r+2}$$

$$= -\frac{(r+2)\,2^{\frac{r}{2}}\,\beta^r\,(\eta+1)^{1-\frac{r}{2}}\,\Gamma\left(\frac{r+2}{2},\frac{\beta^{\frac{2(-r-2)}{r+2}}\,(\eta+1)x^2}{2}\right)}{(r+2)\eta+r+2}+c \quad (19)$$

by simplyfing (19) we get :

$$E(x^r) = -\frac{2^{\frac{r}{2}}\,\Gamma\left(\frac{r+2}{2}\,,\,\frac{(\eta+1)x^2}{2\beta^2}\right)|\beta|^r\,x^r}{(\eta+1)^{\frac{r}{2}}\,|x|^r}+c \quad (20)$$

2-2-2. THE MEAN

The general formula for the first mean of the new weighted Rayleigh distribution about zero is,

$$E(x) = \int_0^{\infty} x f(x)dx$$

$$= \int_0^{\infty} x.(1+\eta)\frac{x}{\beta^2}.e^{-\frac{(\eta+1)\,x^2}{2\beta^2}}dx$$

$$= \frac{(1+\eta)}{\beta^2}\int_0^{\infty} x^2.e^{-\frac{(\eta+1)\,x^2}{2\beta^2}}dx \quad (21)$$

Now, by integrating by parts and simplifying the equation (21), we get the mean:

$$= \frac{\sqrt{\pi}\;|\beta|}{\sqrt{2}\,\sqrt{\eta+1}} \quad (22)$$

2-2-3. THE VARIANCE

The formula for the variance of the new weighted Rayleigh distribution is as follows:

$$v(x) = E(x)^2 - (E(x))^2$$

$$= \frac{2\,\beta^2}{1+\eta} - \left(\frac{\sqrt{\pi}\;|\beta|}{\sqrt{2}\,\sqrt{\eta+1}}\right)^2 \quad (23)$$

2-2-4. THE MOMENT GENERATING FUNCTION

Lemma3

Let x be a random variable of the NWRD; therefore, the moment generating function of the NWRD is

$$\frac{-\Gamma\left(\frac{1}{2},\frac{\left((\eta+1)x-t\beta^2\right)}{2(\eta+1)\beta^2}\right)t\,e^{\frac{t^2\beta^2}{2(1+\eta)}}|\beta|\left((\eta+1)x-t\beta^2\right)}{\sqrt{2}\,\sqrt{\eta+1}\,|(1+\eta)x-t\beta^2|} - \Gamma\left(1,\frac{\left((\eta+1)x-t\beta^2\right)^2}{2(\eta+1)\beta^2}\right)e^{\frac{t^2\beta^2}{2(1+\eta)}}$$

proof

The general form of the moment- generating function of the NWRD is:

$$= \int_0^{\infty} e^{tx}.(1+\eta)\frac{x}{\beta^2}.e^{-\frac{(\eta+1)\,x^2}{2\beta^2}}dx \quad (24)$$

Write,

$$\mathrm{x} = \frac{\beta^2}{-\eta-1}\left(t-\frac{(1+\eta)x}{\beta^2}\right)-\frac{t\beta^2}{-\eta-1} \quad (25)$$

Substitute equation (25) in (24)

$$= \frac{(\eta+1)\,e^{tx-\frac{(\eta+1)x^2}{2\beta^2}}}{-\eta-1} - \frac{\sqrt{\pi}t\sqrt{1+\eta}\beta e^{\frac{t^2\beta^2}{2(1+\eta)}}\;erf\left(((1+\eta)x-t\beta^2)/\sqrt{2}\,\sqrt{1+\eta}\;\beta\right)}{\sqrt{2}\,(-\eta-1)}+c$$

Note $\eta\beta^2+\beta^2>0$

$$= \frac{-\Gamma\left(\frac{1}{2},\frac{\left((\eta+1)x-t\beta^2\right)}{2(\eta+1)\beta^2}\right)t\,e^{\frac{t^2\beta^2}{2(1+\eta)}}|\beta|\left((\eta+1)x-t\beta^2\right)}{\sqrt{2}\,\sqrt{\eta+1}\,|(1+\eta)x-t\beta^2|} - \Gamma\left(1,\frac{\left((\eta+1)x-t\beta^2\right)^2}{2(\eta+1)\beta^2}\right)e^{\frac{t^2\beta^2}{2(1+\eta)}}+c$$

After splitting, apply linearity and after applying the exponential rule:

$\int a^u\,du = \frac{a^u}{\ln(a)}$ with a = e = e^u

And special integral (Gauss error function) erf(u) , we get :

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$$= \frac{1}{-\eta-1}\int\left((-\eta-1)x+t\beta^2\right)e^{tx-\frac{(\eta+1)x^2}{2\beta^2}}dx - \frac{t\beta^2}{-\eta-1}\int e^{tx-\frac{(\eta+1)x^2}{2\beta^2}}dx \quad (26)$$

2-2-5. THE FACTORIA MOMENT GENERATING FUNCTION

this function of the NWRD is proposed as follows:

$$M(t) = E(t^x) = \int_0^\infty t^x\,(1+\eta)\frac{x}{\beta^2}.e^{-\frac{(\eta+1)\,x^2}{2\beta^2}}\,dx$$

$$= \int_0^\infty \frac{t^x(1+\eta)xe^{\frac{-(1+\eta)x^2}{2\beta^2}}}{\beta^2}$$

$$= \frac{-\left(\sqrt{2}\,\Gamma\left(\frac{1}{2},\frac{ln^2(t)\beta^2}{2(1+\eta)}\right)|\ln(t)|\sqrt{1+\eta}\,\beta-2\Gamma\left(1,\frac{ln^2(t)\beta^2}{2(1+\eta)}\right)(1+\eta)e^{\frac{ln^2}{2(}}\right.}{2(1+\eta)} \quad (27)$$

$; 1+\eta > 0 , \beta > 0$

2-2-6. THE MODE

Lemma4

Let x represenst a random variable of the NWRD, the mode of the NWRD is $x = \frac{\beta}{\sqrt{1+\eta}}$

Proof

The mode of the NWRD has been obtained by solving $\frac{dln f_w(x;\beta,\eta)}{dx} = 0$

$$\frac{dln f_w(x;\beta,\eta)}{dx} = \frac{d}{dx}ln\left[(1+\eta)\frac{x}{\beta^2}.e^{-\frac{(\eta+1)\,x^2}{2\beta^2}}\right]$$

$$= \frac{d}{dx}ln\left[\frac{(1+\eta)}{\beta^2}x\,e^{-\frac{(\eta+1)\,x^2}{2\beta^2}}\right] \quad (28)$$

Let

$$V^2 = \frac{\beta^2}{(1+\eta)} \quad (29)$$

Substitute (29) in (28) , then

$$\frac{d}{dx}ln\left[\frac{x}{V^2}\ .e^{-\frac{x^2}{2V^2}}\right] = 0$$

$$\frac{1}{x} - \frac{x}{V^2} = 0$$

$$x^2 = V^2$$

After Simplified,will get the mode at $x = x_o$

$$x_0 = \frac{\beta}{\sqrt{1+\eta}}$$

2-2-7. THE MEDIAN

Lemma5

Let x be a r.v. of the NWRD, then the median of the NWRD is, $x = \beta\sqrt{\frac{2ln2}{1+\eta}}$

proof

The median of the new weighted Rayleigh distribution is expressed as:

$$1 - e^{-\frac{(\eta+1)\,x^2}{2\beta^2}} = \frac{1}{2} \quad (30)$$

Now, solve F(x) = $\frac{1}{2}$. By substituting (29) in (30)

$$e^{-\frac{x^2}{2V^2}} = \frac{1}{2}$$

$$\frac{x^2}{2V^2} = ln2$$

Then

$$x = V\sqrt{2ln2} \quad (31)$$

Undo, substitute the value of v in (31) , then $x = \beta\sqrt{\frac{2ln2}{1+\eta}}$, Since x is greater than zero, its negative value will be disregarded.

2-2-8. COEFFICIENT OF SKEWNESS

Lemma 6

Let x be a r.v. of the NWRD, then the Coefficient of Skewness of NWRD is :

$$\frac{\frac{\sqrt[3]{\pi}\,(1+\eta)}{8\beta^2\left(\frac{\eta}{2\,\beta^2}+\frac{1}{2\,\beta^2}\right)^{5/2}}}{\left(\frac{2\,\beta^2}{1+\eta}\right)^{3/2}}$$

proof

The standard expression for the coefficient of skewness (C.S) of the new weighted Rayleigh distribution is:

$$C.R_{WR} = \frac{E(x^3)_{WR}}{(E(x^2)_{WR})^{3/2}}$$

$$= \frac{\frac{\sqrt[3]{\pi}\,(1+\eta)}{8\beta^2\left(\frac{\eta}{2\,\beta^2}+\frac{1}{2\,\beta^2}\right)^{5/2}}}{\left(\frac{2\,\beta^2}{1+\eta}\right)^{3/2}} \quad (32)$$

The proposed model reaches shape flexibility over parameter decomposition , while the competing models often achieve skewness through adding mathematical complexity .

2-2-9. COEFFICIENT OF KURTOSIS

Lemma7

Let x be a r.v. of the NWRD, then the Coefficient of kurtosis of the NWRD is $\frac{\frac{8\,\beta^4}{(1+\eta)^2}}{\left(\frac{2\,\beta^2}{1+\eta}\right)^2} - 3$

proof

The standard expression for the coefficient of kurtosis (C.K) of the new weighted Rayleigh distribution is:

$$C.R_{WR} = \frac{E(x^4)_{WR}}{(E(x^2)_{WR})^2} - 3$$

$$C.R_{WR} = \frac{\frac{8\,\beta^4}{(1+\eta)^2}}{\left(\frac{2\,\beta^2}{1+\eta}\right)^2} - 3 \quad (33)$$

2-2-10. QUNTAILE FUNCTION

Lemma 8

Let x be a r.v. of the NWRD, then the Quantile function of the NWRD is $\sqrt{-\frac{2\beta^2}{(\eta+1)}\ln(1-Q)}$

proof

The following solution derives the quantile function for NWRD :

$F(x_{(Q)}) = Pr(X \le x_{(Q)})$

Now, by applying the inverse transformation to F(x) in equation (11) :

$x_{(Q)} = F^{-1}(Q)\,; \quad x_{(Q)} > 0\,, 0 < Q < 1$

$Q = 1 - e^{-\frac{(\eta+1)\,x^2}{2\beta^2}}$

$\ln(1-Q) = -\frac{(\eta+1)\,x^2}{2\beta^2}$

$$x = \sqrt{-\frac{2\beta^2}{(\eta+1)}\ln(1-Q)}\,,\ \beta > 0, \eta+1 > 0\,,\ 0 \le Q < 1 \quad (34)$$

2-2-11. CHARACTERISTIC FUNCTION

Lemma9

Let x be a r.v. of the NWRD, then , the NWRD's characteristic function is proposed as

Proof

The standard representation of the characteristic function for the new weighted Rayleigh distribution is:

$E(e^{itx}) = \int_0^\infty e^{itx}.(1+\eta)\frac{x}{\beta^2}.e^{-\frac{(\eta+1)\,x^2}{2\beta^2}}\,dx$

$$= \frac{(1+\eta)}{\beta^2}\int_0^\infty x\,e^{itx-\frac{(\eta+1)\,x^2}{2\beta^2}}\,dx \quad (35)$$

Now solving the integral part of (35)

$$\int_0^\infty x\,e^{itx-\frac{(\eta+1)\,x^2}{2\beta^2}}\,dx \quad (36)$$

Write

$$x = \frac{it\beta^2}{\eta+1} - \frac{\beta^2}{\eta+1}\left(it - \frac{(1+\eta)x}{\beta^2}\right) \quad (37)$$

Substituted the (37) in (36) , we get

$$\int_0^\infty \left(\frac{it\beta^2 e^{itx-\frac{(\eta+1)\,x^2}{2\beta^2}}}{\eta+1} - \beta^2\left(it - \frac{(1+\eta)x}{\beta^2}\right)e^{itx-\frac{(\eta+1)\,x^2}{2\beta^2}}\right)dx \quad (38)$$

Now apply (38) in (35), will get :

$$\frac{1}{\eta+1}\int_0^\infty ((\eta+1)x - it\beta^2)\,e^{itx-\frac{(\eta+1)\,x^2}{2\beta^2}}\,dx + \frac{it\beta^2}{\eta+1}\int_0^\infty e^{itx-\frac{(\eta+1)\,x^2}{2\beta^2}}\,dx \quad (39)$$

Now solving the first integral of (39)

$$\int_0^\infty ((\eta+1)x - it\beta^2)\,e^{itx-\frac{(\eta+1)\,x^2}{2\beta^2}}\,dx \quad (40)$$

Let

$$u = itx - \frac{(\eta+1)\,x^2}{2\beta^2} \quad (41)$$

Then

$du = it - \frac{(\eta+1)\,x}{\beta^2}\,dx$

$$= -\beta^2\int_0^\infty e^u du \quad (42)$$

Now solving the integral part of (42)

$$\int_0^\infty e^u du \quad (43)$$

Apply exponential rule on (43):

$\int_0^\infty a^u du = \frac{au}{ln(a)}$, when a = e = e^u put it in (42)

$= -\beta^2 e^u$

By Substitute $u = \frac{(\eta+1)\,x - it\beta^2}{\sqrt{2}\,\beta\sqrt{\eta+1}}$, $du = \frac{\sqrt{\eta+1}}{\sqrt{2}\,\beta}\,dx$, to get

$= \frac{\sqrt{\pi}\,\beta e^{\frac{-t^2\beta^2}{2(\eta+1)}}}{\sqrt{2}\,\sqrt{\eta+1}}\int_0^\infty \frac{2\,e^{-u^2}}{\sqrt{\pi}}\,du$ ...(43)

Now solving the integral part of (43)

This is a special integral (guass erroe function)= erf(u)

Put it in equation (43) with substitute u

$$= \frac{\sqrt{\pi}\ \beta e^{\frac{-t^2\beta^2}{2(\eta+1)}}\,erf\left(\frac{(\eta+1)\,x - it\beta^2}{\sqrt{2}\,\beta\sqrt{\eta+1}}\right)}{\sqrt{2}\,\sqrt{\eta+1}} \quad (44)$$

$$= \frac{\sqrt{\pi}\, it\beta^3\, e^{\frac{-t^2\beta^2}{2(\eta+1)}}\, erf\left(\frac{(\eta+1)\,x - it\beta^2}{\sqrt{2}\,\beta\sqrt{\eta+1}}\right)}{\sqrt{2}\,(\eta+1)^{3/2}} - \frac{\beta^2 e^{itx-\frac{(\eta+1)\,x^2}{2\beta^2}}}{\eta+1} \quad (45)$$

$$= \frac{\sqrt{\pi}\, it\beta\, e^{\frac{-t^2\beta^2}{2(\eta+1)}}\, erf\left(\frac{(\eta+1)\,x-it\beta^2}{\sqrt{2}\,\beta\sqrt{\eta+1}}\right)}{\sqrt{2}\,(\eta+1)} - e^{itx-\frac{(\eta+1)\,x^2}{2\beta^2}} + c \quad (46)$$

, assumed that $\beta^2\eta + \beta^2 > 0$ ...(46)

After simplifying and substituting in (35), will get :

$$= -\frac{\Gamma\left(\frac{1}{2},\frac{\left((\eta+1)x-it\beta^2\right)^2}{2(\eta+1)\beta^2}\right) t\,|\beta|\,e^{\frac{-t^2\beta^2}{2(\eta+1)}}\,(i(\eta+1)x+t\beta^2)}{\sqrt{2}\,(\eta+1)\sqrt{\frac{\left((\eta+1)x-it\beta^2\right)^2}{\eta+1}}} - \Gamma\left(1,\frac{\left((\eta+1)x-it\beta^2\right)^2}{2(\eta+1)\beta^2}\right) e^{\frac{-t^2\beta^2}{2(\eta+1)}} + c$$

2-3. MAXIMUM LIKELIHOOD ESTIMATION

In this part of the article, the parameters of the new weighted Rayleigh distribution have been determined employing the maximum likelihood estimation technique (Ali & Al Kanani, 2021). The idea of this method is to maximize the parameters for the likelihood function.

Consider n is the size of the random sample, which contains values

$x_1, x_2, \ldots, x_n$ from the weighted Rayleigh density.

The likelihood function of the NWRD function is:

$$l(x;\beta,\eta) = \prod_{i=1}^{n}(1+\eta)\frac{x}{\beta^2}.e^{-\frac{(\eta+1)\,x^2}{2\beta^2}}$$

$$= (1+\eta)^n.\left(\frac{1}{\beta^2}\right)^n.\prod_{i=1}^{n} x_i.\, e^{-\frac{(\eta+1)\,\sum_{i=1}^{n} x^2}{2\beta^2}}$$

The log likelihood function will now be as follows:

$$\ln l\,(x;\beta,\eta) = n\,ln(1+\eta) - 2nln\beta + \sum_{i=1}^{n}\ln x_i - \frac{(\eta+1)}{2\beta^2}\sum_{i=1}^{n}(x_i)^2 \quad (47)$$

Now, let's denote:

$$O = \sum_{i=1}^{n}(x_i)^2 \quad (48)$$

$$Z = \sum_{i=1}^{n}\ln x_i \quad (49)$$

Then, when substitute (48) and (49) in (47) ,will get :

$$l(\beta,\eta) = nln(1+\eta) - 2ln\beta + z - \frac{(\eta+1)}{2\beta^2}\,o \quad (50)$$

The log-likelihood function (50) and its first partial derivatives with regard to β and η are as follows:

$$\frac{\partial l}{\partial\beta} = \frac{-2n}{\beta} + \frac{1+\eta}{\beta^3}\,o \quad (51)$$

$$\frac{\partial l}{\partial\eta} = \frac{n}{1+\eta} - \frac{1}{2\beta^2}\,o \quad (52)$$

Now, after setting the equations (51) and (52) to zero as follows:

$$\frac{-2n}{\beta} + \frac{1+\eta}{\beta^3}\,o = 0$$

$$\frac{n}{1+\eta} = \frac{1}{2\beta^2}\,o$$

And solving them simultaneously yields the two parameters' maximum likelihood estimations. By applying the second partial derivatives of equations (51) and (52), the matrix of Fisher information can be formulated by deriving the negative expectations of the aforementioned second derivatives.

The maximum likelihood estimators' variance and covariance matrix is denoted as Fisher's matrix. The likelihood function of the proposed model is a unique maximum ridge .

2-4. APPLICATION

Here in the article, the application of the NWRD is acquired through the utilization of the
lifetime dataset of 23 ball bearings (see (Lieblein & Zelen, 1956)). These datasets are using the distribution. The data were 23 balls (in 12) but in this study, the data has been chosen as C.Carony in his article (see (Caroni, 2002)).

In this article, the weighted Rayleigh distribution is fitted to this data, and the results are compared to those of (13) . The statistics revealed that the new weighted Rayleigh distribution aligns more accurately with the data than the two-parameter Rayleigh distribution and the Weibull distribution in (13). See table (1) .

X_i=152.7,172.0,172.5,173.3,193.0,204.7,216.5,234.9, 262.6,422.6,500.5,524.8,527.8,622.3,642.2,782.5,864 .1,1100.3,1304.7,527.8 .

Choose β = 156

η = - 0.770

calculate log-likelihood, AIC (Akaike Information Criterion) , BIC (Bayesian Information Criterion)

When comparing with the Rayleigh distribution (RD) and Weibull distribution (WD) in (13) we get :

TABLE 1. comparing of the NWRD with RD and WD

Distribution	Log-likelihood	AIC	BIC
NWRD	-107.118	218.236	220.125
RD	-109.53	221.06	222.95
WD	-107.43	218.86	220.75

2-5. The CONCLUSIONS

This work introduces a new weighted Rayleigh distribution based on Gupta and Gundu modified weighted according to Azzalini's idea. Where the Azzalini-type proposed a way to variation the essential asymmetry of the distributions. The proposed distribution proposed a flexible way to alter the scaling by a weighted constant. Unlike standard models that proposed compound transcendental, the proposed probability density function novelty deceits in its structural decomposition for the Rayleigh scale, using a weighted distribution framework to clearly separate the basic population scale from the external weighting bias.

The proposed model basically depends on the Azzalini (1985) skew-symmetric idea. We donot only add a parameter randomly; we tracked the world-standard way for proposing skewness (Azzalini) and enhanced it specifically for the Rayleigh.

The new distributions' most important properties have been derived.

This new weighted distribution's implementation has been used on the ball bearing dataset, and the outcomes have been contrasted with other studies using the lifetime experiment.

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