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Neutrosophic Modeling of Lifetime Data Using the Weibull–Exponentiated Exponential Distribution

Ayat Khaled Sagheer [a*]

[a] *Department of Mathematics, College of Education for Pure Sciences, University of Kerbala*

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Abstract

Modeling lifetime data plays an indispensable role in reliability engineering and survival analysis, where the primary goal is to accurately predict time-to-event phenomena. While classical probability distributions, such as the widely applied Weibull-Exponentiated Exponential (W-EE) hybrid model, offer significant flexibility in capturing versatile hazard rates, they possess a fundamental limitation. Specifically, traditional models inherently assume that the observed data and underlying parameters are precise and well-defined. However, this assumption is rarely satisfied in practical scenarios. In real-world applications, data collection is frequently plagued by measurement errors, interval-censored information, environmental noise, and inherent vagueness. Relying strictly on classical statistical methods under such uncertain conditions often yields biased parameter estimates and misleading reliability inferences. To address these critical challenges, this study introduces the Neutrosophic Weibull–Exponentiated Exponential (NW–EE) distribution, a novel extension designed to model lifetime data characterized by indeterminacy. By integrating neutrosophic components into the classical W-EE model, we effectively represent observations and parameters as neutrosophic intervals rather than exact point values. In this paper, we rigorously derive the key statistical functions of the new distribution, including the neutrosophic cumulative distribution function (NCDF), the neutrosophic probability density function (NPDF), and vital reliability measures such as the survival and hazard rate functions. Furthermore, we investigate essential mathematical properties such as neutrosophic moments and variance. Additionally, the model parameters are estimated utilizing the neutrosophic maximum likelihood estimation (NMLE) technique, and the model is validated using real datasets. The results demonstrate that the NW-EE model offers superior flexibility in handling imprecise observations in reliability engineering and survival analysis, providing a highly accurate and realistic alternative to classical distributions. The introduced Neutrosophic Weibull-Exponentiated Exponential (NW-EE) distribution is a sophisticated analytical tool designed to model lifetime data in environments where imprecision and indeterminacy are unavoidable.

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1. INTRODUCTION[1]

Lifetime data modeling and survival analysis are fundamental pillars in reliability engineering, medical survival analysis, and risk assessment. The primary objective in these fields is to accurately model time-to-event data, such as the failure time of mechanical components or the survival time of patients. This requires highly flexible probability distributions capable of accommodating diverse hazard rate behaviors, including constant, monotonically increasing, decreasing, and non-monotonically increasing, decreasing, and non-monotonic shapes. These foundational concepts have been extensively studied by pioneers in survival analysis and reliability, such as (Wayne B. Nelson ,2004), (John f. lawless,2003), (William Q. Meeker and L. A. Escobar ,1998).

Historically, the standard probability models were limited in handling complex failure mechanisms. To overcome this, the Weibull distribution, originally introduced by (Waloddi Weibull ,1951), emerged as a powerful model and has been extensively applied in various engineering contexts (Robert B. Abernethy,2006). Despite its widespread use, the traditional Weibull model struggles with complex, non-

*Corresponding Author Institutional Email: ayat.khaled@uokerbala.edu.iq (Ayat Khaled Sagheer)

monotonic failure rates consequently, (R. D. Gupta and D. Kundu,1999) proposed the Exponentiated Exponential (EE) distribution as a robust extension, providing additional flexibility over the standard Exponential distribution.

To achieve even greater modeling adaptability, subsequent research explored various extensions and generalization of these base distributions. Notable contribution includes the comprehensive survey by (Seralees Nadarajah and S. Kotz,2006) , as well as the generalized Weibull-G family framework introduced by (Gauss M. Cordeiro et al., 2013). Integrating these concepts led to hybrid models, for instance, the foundational characteristics and classical application of the Weibull-Exponentiated Exponential (W-EE) distribution were detailed by (A. K. S. Hamad,2021) in her comprehensive study on lifetime data modeling. This hybrid model leverages the strengths of both parent distribution to capture versatile hazard rate shapes that standard base models cannot adequately explain.

However, a significant limitation of these sophisticated classical models is their fundamental reliance on the assumption that the observed data are crisp, precise, and perfectly determined. In real-world practical environments, lifetime data are rarely exact. They frequently involve measurement errors, incomplete observations, censored or vague information, and environmental uncertainty. Under such circumstances, relying solely on classical statistics frequently results in skewed parameter estimates, false reliability inferences, and poor decision-making. These difficulties need for sophisticated modeling techniques that can specifically account for imprecision and indeterminacy.

To address these complex issues, the Neuromorphic framework was originally introduced by Florentin Smarandache as a powerful generalization of existing fuzzy logic theories. Unlike traditional approaches, Neutrosophic logic independently quantifies the degree of truth, falsity, and indeterminacy, allowing it to incorporate indeterminacy explicitly into the probabilistic structure. Building upon this, recent groundbreaking development in Neutrosophic statistics spearheaded by researchers like (Muhammad Aslam ,2018) have demonstrated the high effectiveness of this approach in handling uncertain data across various application. In this framework, parameters and observation are mathematically represented as Neutrosophic intervals rather than exact point values, providing a much more realistic representation of physical phenomena.

Motivated by these advancements and the critical need to model complex imprecise lifetime data, this paper develops a Neutrosophic generalization of the Weibull-Exponentiated Exponential distribution.

The man’s contributions of this paper are:

- Definition of the Neutrosophic Weibull-Exponentiated Exponential (NW-EE) distribution.
- Derivation of its Neutrosophic cumulative distribution function (NCDF) and Neutrosophic probability density function (NPDF)
- Development of the core statistical and mathematical properties of the proposed model.
- Construction of Neutrosophic reliability measures to evaluate system performance under uncertainty

2. METHODOLOGY

Let X be a continuous Neutrosophic random variable representing lifetime as $X_N = x + I$.

Definition 1: The Cumulative distribution function of Weibull–Exponentiated Exponential Distribution (W-EE) is

$F(x) = 1 - e^{-ax^b(1-e^{-\alpha x})^\lambda}$, $x > 0$ where

$a > 0$ is the scale parameter

$b > 0$ is the Weibull shape parameter

$\alpha > 0$ is the exponential scale parameter

$\lambda > 0$ are the exponentiation parameters

Definition 2: The Neutrosophic Probability Density Function of Exponential Distribution (NPDF) is

$f_{x_N(x)=} f(x - I) = \lambda e^{-\lambda(x-I)}, x > I$

The Cumulative distribution function of Neutrosophic Exponential Distribution (NCDF) is

$F_{x_N}(x - I) = \int_0^x f_{xN}(x) = \int_0^x \lambda e^{-\lambda(x-I)} = e^{-\lambda I}\left(1 - e^{-\lambda x}\right), \; x \geq I$

Definition 3: The Cumulative distribution function of Neutrosophic Weibull- Exponentiated Exponential Distribution (NW-EE) is:

$F_{x_N}\big((x-I),a,b,\alpha,\lambda\big)=1-e^{-a(x-I)^b[1-e^{-\alpha(x-I)}]^\lambda}$, $x\geq I$

Definition 4: The Neutrosophic Probability Density Function of Neutrosophic Weibull– Exponentiated Exponential Distribution (NW-EE) is:

$$f_{x_N}\big((x-I),a,b,\alpha,\lambda\big)=a(x-I)^b e^{-a(x-I)^b[1-e^{-\alpha(x-I)}]^\lambda}.[1-e^{-\alpha(x-I)}]^\lambda.[\alpha\lambda e^{-\alpha(x-I)}+\frac{b}{x-I}]$$

The Neutrosophic Reliability Function:

$$R_{x_N}\big((x-I),a,b,\alpha,\lambda\big)=1-F_{x_N}\big((x-I),a,b,\alpha,\lambda\big)$$

The Neutrosophic Hazard function:

$H_{x_N}\big((x-I),a,b,\alpha,\lambda\big)=\frac{f_{x_N}((x-I),a,b,\alpha,\lambda)}{R_{x_N}((x-I),a,b,\alpha,\lambda)}$, the hazard function rate can exhibit increasing, decreasing, or bathtub shapes under indeterminacy.

Definition 5: The limits of the NPDF and NCDF.
The Nw-EE distribution exhibits specific and important behavior for its Neutrosophic probability density function(Npdf) and Neutrosophic cumulative distribution function (Ncdf) as the neutrosophic random variable $(X-I)$ approaches the boundaries of its domain , namely zero and infinity mathematical analysis of these limits confirms that the functions tend towards zero which is consistent with the fundamental properties of continuous neutrosophic probability distributions.
The asymptotic behavior of the Npdf is formally described by the following limits:

1. As$(x-I)$ approaches $o(x-I)\to 0)$ the value of Npdf tends to zero. This is mathematically expressed as:

$$\lim_{(x-I)\to 0} f_{xN}\big((x-I),a,b,\alpha,\lambda\big)=0$$

This result is derived from the fact that the constituent terms within the Npdf , such as $(x-I)^b$ and $\big[1-e^{-\alpha(x-I)}\big]^\lambda$, approach zero, causing the entire expression to vanish.

2. As$(x-I)$ approaches infinity $((x-I)\to\infty)$ the value of Npdf also tends to zero, as shown in the following equation

$$\lim_{(x-I)\to\infty} f_{xN}\big((x-I),a,b,\alpha,\lambda\big)=0$$

In this case, although some terms may increase the dominant factor is the negative exponential term $e^{-a(x-I)^b[1-e^{-\alpha(x-I)}]^\lambda}$.

This term ensures the decay of the Npdf to zero for larg value of $(x-I)$. This behaviour, where the Npdf approaches zero at both ends of the support, is a characteristic feature of many continuous neutrophic probability distribution. It confirms that the distribution is unimodal and that the neutrosphic probability mass is concentrated in a finite region.

Definition 6: Limit of Neutrosophic cumulative distrbuion function (Ncdf)
The behaviour of neutrosophic cumulative distribution function at the boundaries reflects another fundamental property, namely that its value ranges from zero to one

1. As $(x-I)$ approaches $o(x-I)\to 0)$: the Ncdf correctly tend to zero. This s indicates that the probability of the neutrosopic random variable X_N taking a value less than or equal to zero which is appropriate for a distribution defined on the positive real line. This limit is given by

$$\lim_{(x-I)\to 0} F_{xN}\big((x-I),a,b,\alpha,\lambda\big)=\lim_{(x-I)\to 0} e^{-a(x-I)^b[1-e^{-\alpha(x-I)}]^\lambda}=0$$

In summary, this limit analysis confirms the validity of its NW-EE distribution, the Npdf correctly converges to zero at the extremes of its support, and the Ncdf correctly converges to zero zero at the lower bound of its domain

Definition 7: The Neutrosophic mean o (NW-EE) is obtained by

$$E(X_N)=\int_I^\infty x f_{xN}((x-I),a,b,\alpha,\lambda)\,dx$$

The Neutrosophic r^{th} moment about the origin is defined by $\acute{M}_r=\int_I^\infty x^r f_{xN}((x-I),a,b,\alpha,\lambda)\,dx$
Substituting the (NW-EE) distribution gives $\acute{M}_r=\int_I^\infty x^r a(\lambda x)^b e^{-a(x-I)^b[1-e^{-\alpha(x-I)}]^\lambda}.\Big[\frac{\alpha\lambda}{e^{\alpha(x-I)}}+\frac{b}{x-I}\Big]\,dx$
The above expectation represents the general neutrosophic moment of order r

This represents the average lifetime of system under uncertainty and the variance measurres dispersion around the mean defined by:
$var(X_N)=E(X^2{}_N)-\big(E(XN)\big)^2$ where

$$E(X^2{}_N) = \int_I^\infty x^2 f_{xN}((x-I),a,b,\alpha,\lambda)\,dx$$

Both integrals converge due to the exponent decay term in the density function and the moment generated function of the neutrosophic random variable is $M_{XN}(t) = E(e^{txN}) = \int_I^\infty e^{txN} f_{xN}((x-I),a,b,\alpha,\lambda)\,dx$ and differentiating the $M.g.f$ yields the moments of the distribution.

Definition 8: Let $x_1, x_2, \dots x_n$ be a random sample the likelihood function is $L = \prod_{i=1}^n f_{xN}((x-I),a,b,\alpha,\lambda)$ and taking logarithms, we obtain

$$l = \sum_{i=1}^n ln f_{xN}((x-I),a,b,\alpha,\lambda)$$

The parameter estimation is obtained by solving

$$\frac{\partial l}{\partial a} = 0\,, \frac{\partial l}{\partial \lambda} = 0\,, \frac{\partial l}{\partial \alpha} = 0\,, \frac{\partial l}{\partial \lambda} = 0$$

Definition 9: The Neutrosophic cumulative Hazard function plays an important role in reliability analysis and survival studies. It measures the accumulated risk of failure up to a given time $H_N\big((x-I),a,b,\alpha,\lambda\big) = -\ln[1 - F_N(x-I)]$

Substituting the Neutrosophic cumulative distribution function gives

$$H_N\big((x-I),a,b,\alpha,\lambda\big) = -\ln\left[e^{-a(x-I)^b\left[1-e^{-\alpha(x-I)}\right]^\lambda}\right]$$

Thus, the Neutrosophic cumulative Hazard function becomes

$$H_N\big((x-I),a,b,\alpha,\lambda\big) = a(x-I)^b\left[1-e^{-\alpha(x-I)}\right]^\lambda$$

Note that the Neutrosophic cumulative hazard function presents the accumulated failure risk of the system while incorporating ideterminacy in the lifetime observation.

The parameter I reflects the uncertainty associated with the measurement or observation of the lifetime variable when $I = 0$, the neutrosophic model reduce to the classical cumulative hazard function.

Definition 10: The Neutrosophic order statistics. Order statistics are crucial in reliability of studying the lifetime of a system with n components.

Let $X_{N(1)}, X_{N(2)}, \dots, X_{N(n)}$ be a Neutrosophic random sample from the NW-EE distributions.

The N.PDF of the K^{th} order statistics $f^N{}_{K:N}(x-I)$ is given by

$$f^N{}_{K:N}(x-I) = \frac{n!}{(K-1)!(n-K)!} f_{x_N}(x-I)\left[F_{x_N}(x-I)\right]^{K-1}\left[1 - F_{x_N}(x-I)\right]^{n-K}$$

by substitution the N.CDF and N.PDF of our NW-EE distribution , we obtain:

$$f^N{}_{K:N}(x-I) = \frac{n!}{(K-1)!(n-K)!} a(x-I)^b e^{-a(x-I)^b\left[1-e^{-\alpha(x-I)}\right]^\lambda}$$

Definition 11: Neutrosophic Moments of residual life. The mean residual life (MRL) is vital measure in survival analysis. For the NW-EE distribution, the Neutrosophic MRL function denoted by $M_N(t)$ represents the expected remaining life of a component given it has survived up to time $t + I$:

$$M_N(t) = E[X_N - t | X_N > t] = \frac{1}{R_N(t-I)} \int_t^\infty R_N(t-I)dx$$

Substituting the Neutrosophic reliability function $R_N(t-I)$:

$$M_N(t) = \frac{\int_t^\infty e^{-a(x-I)^b\left[1-e^{-\alpha(x-I)}\right]^\lambda}}{e^{-a(x-I)^b\left[1-e^{-\alpha(x-I)}\right]^\lambda}}$$

The stress – strength reliability analysis $R_N = P(Y_N < X_N)$ in engineering, we often evaluate the probability that a system's strength X_N exceeds the stress Y_N that is applied to it.

If the X_N and Y_N follow the $NW-EE$ distribution with different of scale parameters a_1, a_2, the Neutrosophic stress– strength reliability is

$$R_{NS} = \int_I^\infty f_{N1}(X-I,a_1,b,\alpha,\lambda)\, f_{N2}(X-I,a_2,b,\alpha,\lambda)dx$$

Definition 12: Computational estimation using Neutrosophic maximum likelihood (NMLE), to estimate the unknown parameter $\theta = (a,b,\alpha,\lambda)$ of the NW-EE distribution, we employ the Neutrosophic maximum likelihood estimation (NMLE) method.

Given a Neutrosophic random sample

$X_{N(1)}, X_{N(2)}, \dots, X_{N(n)}$ the likelihood function is defined as the product of Neutrosophic probability density function

$L(\theta) = \prod_{i=1}^{n} f_{x_N}((X_i - I), a, b, \alpha, \lambda)$

To simplify the estimation, we take the natural logarithm to obtain the log-likelihood function

$$l(\theta) = n \ln a + b \sum_{i=1}^{n} \ln(X_i - I) - a \sum_{i=1}^{n} (X_i - I) \left[1 - e^{-\alpha(X_i - I)}\right]^{\lambda} + \cdots$$

Where the likelihood equations of the parameter estimates are obtained by solving the system of non-linear equations generated by taking the partial derivatives of $l(\theta)$ with respect to each parameter and setting them to zero

1. For parameter a :

$$\frac{\partial l}{\partial a} = \frac{n}{a} - \sum_{i=1}^{n} (X_i - I)^b \left[1 - e^{-\alpha(X_i - I)}\right]^{\lambda} = 0$$

2. For parameter α :

$$\frac{\partial l}{\partial \alpha} = -a \sum_{i=1}^{n} (X_i - I)^b \lambda \left[1 - e^{-\alpha(X_i - I)}\right]^{\lambda - 1} . \alpha\lambda e^{-\alpha(X_i - I)} = 0$$

Conclusion: This study has successfully introduced the Neutrosophic Weibull-Exponentiated Exponential (NW-EE) distribution as a sophisticated analytical tool designed to model lifetime data in environments where imprecision and indeterminacy are unavoidable.

By explicitly incorporating the Neutrosophic parameter I into the classical Weibull-Exponentiated Exponential framework, the model provides a superior level of flexibility.

In summary the NW-EE distribution offers a more realistic and comprehensive framework than its classical counterparts for analyzing lifetime data under uncertainty. Further research should focus on extending this model to include multivariate Neutrosophic cases and conditions in comparative studies with real-world datasets in medical and industrial reliability testing.

REFERENCES

Waloddi Weibull, "A statistical distribution function of wide applicability," Journal of Applied Mechanics, vol. 18, pp. 293–297, 1951.

R. D. Gupta and D. Kundu, "Generalized exponential distribution," Australian & New Zealand Journal of Statistics, vol. 41, no. 2, pp. 173–188, 1999.

Wayne B. Nelson, Applied Life Data Analysis, New York: Wiley, 2004.

John F. Lawless, Statistical Models and Methods for Lifetime Data, 2nd ed., New York: Wiley, 2003.

William Q. Meeker and L. A. Escobar, Statistical Methods for Reliability Data, New York: Wiley, 1998.

Saralees Nadarajah and S. Kotz, "The exponentiated exponential distribution: A survey," AStA Advances in Statistical Analysis, vol. 90, pp. 1–14, 2006.

Gauss M. Cordeiro, E. M. M. Ortega, and D. C. C. da Cunha, "The Weibull-G family of distributions," Journal of Statistical Computation and Simulation, vol. 83, no. 7, pp. 1225–1245, 2013.

A. K. S. Hamad, "Modeling for the Lifetime Data with Applications," M.S. thesis, Dept. Math., Coll. Educ. Pure Sci., Univ. Babylon, Babylon, Iraq, 2021.

Robert B. Abernethy, The New Weibull Handbook, 5th ed., North Palm Beach, FL: Robert B. Abernethy, 2006.

Florentin Smarandache, Neutrosophy: Neutrosophic Probability, Set, and Logic, Rehoboth: American Research Press, 1998.

Florentin Smarandache, Introduction to Neutrosophic Statistics, Craiova: Sitech & Education Publishing, 2014.

Muhammad Aslam, "Neutrosophic statistical process control charts," Symmetry, vol. 10, no. 9, p. 395, 2018.

M. Aslam, "Neutrosophic statistics: Theory and applications," Journal of Intelligent & Fuzzy Systems, vol. 36, pp. 1–12, 2019.